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ASPECTS OF THE NUCLEON-NUCLEON INTERACTION

BY



JACOB DE KAM

A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled ASPECTS OF THE NUCLEON-NUCLEON INTERACTION submitted by Jacob de Kam in partial fulfilment of the requirements for the degree of Master of Science in Theoretical Physics.

SUMMARY

This review deals with aspects of the nucleon-nucleon interaction below the pion production threshold. We are mainly concerned with the nucleon-nucleon phase parameters and the predictions made for these by the various meson theoretical models of the nuclear force. In several of these models the predictions are represented in the form of a potential, which is compared with a successful phenomenological potential. In still a few other cases, predictions are made directly for the observables.

In chapter I the phase parameters, phenomenological potentials, observables and their relation are briefly discussed.

Chapter II is devoted to a few standard techniques in meson field theory. Attention is given mainly to the covariant perturbation theory (Feynman rules) and the dispersion relations. These are the two basic techniques used in the various meson theoretical models.

In the chapters III and IV a number of these models is reviewed. In chapter III we consider models in which only the lowest order contributions of the meson exchanges are included. More general models are discussed in chapter IV.

We shall reach the conclusion that present day meson theory is able to describe the nuclear interaction adequately for the F - and higher partial waves. A satisfactory description for the S - and P - waves is possible, but involves the use of form factors. These form factors are not free from arbitrariness. The D -waves are not described well in most models.

PREFACE

"The intrinsic value of the two nucleon problem is reflected by the fact that it has been intensively researched by a huge army of physicists all over the world." (Moravscik : reference 1)

Golberger writes in 1960 : "It is also true that scarcely ever has the world of physics owed so little to so manyIn general, in surveying the field, one is oppressed by the unbelievable confusion and conflict that exists. It is hard to believe that many of the authors are talking about the same problem, or in fact that they know what the problem is." (quoted in reference 21) This depressing state of affairs lasted until : "The discovery was made that the nucleon-nucleon problem could not really be solved without prior solution of the pion-nucleon - and even pion-pion problems." (Signell : reference 20) And in 1972 Moravscik states : "The nucleon-nucleon problem has proven a challenging, novel, rich and fascinating problem and, as a considerable amount of insight has been gained into it, an aesthetically satisfying picture has emerged." (ref. 1)

Although since the recent developments concerning quarks, Q.C.D. and gauge field theories one may hesitate to underline Moravscik's opinion : "The two nucleon interaction is perhaps the most central problem in modern physics" (ref. 1), one may still agree with Breit, who called the two nucleon interaction "the Mount Everest of nuclear physics" (quoted in reference 3). One may gather from these quotations that the two nucleon problem is a worth-while object of study. It is also clear that the problem has many aspects. In this review only a number of these are discussed. First of all the nuclear force

was studied for energies below the pion production only. This regime is most interesting from a nuclear physics point of view. Important related subjects which are not or only marginally covered in this review are :

the experimental state of affairs, many-body applications, the electro-magnetic and weak interactions between nucleons, the low energy behaviour and the deuteron problem, boundary condition models, soft pion theorems and chiral symmetry, intermediate ρ , strong ρ - and non-linear coupling theories, Regge theory, dual theories and the Veneziano model and applications of the Padé approximants technique.

I had the benefit of many other reviews that exist on the various aspects of the field. Especially useful was the article by Moravscik (ref. 1), which is an encyclopedic summary of almost all developments during the period 1959 - 1971 and the very extensive review articles in the 'Supplements of Progress of Theoretical Physics' (ref. 69). Review articles emphasizing particular aspects are the references 2, 3, 4, 20, 21, 23 and 31. Reference 2 treats the theory of the empirical analysis. In reference 3 One Boson Exchange models are considered. Special attention to Two Pion Exchange effects is given in reference 4. Reference 20 covers nuclear potentials. Review of older methods (before 1960) are given in the references 21, 23 and 31.

I have tried to make a compromise between a detailed treatment such as in the Supplements of Theoretical Progress and an encyclopedic approach as given by Moravscik.

The chapters of this review are rather self-contained. In the introductions a brief summary is given of the different sections. Furthermore each chapter closes with a brief discussion of the results.

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CHAPTER I

THE PHENOMENOLOGICAL ANALYSIS

1) Introduction

Experimental information on the interaction between two nucleons can be obtained from reactions involving free nucleons. Another possible source of information is the study of the bound state of the two-nucleon system, the deuteron. Furthermore knowledge may be extracted from many-nucleon systems such as nuclei. However the overwhelming fraction of the quantitative information about the nuclear force originates from scattering experiments involving two nucleons (ref. 1). The deuteron has contributed little to the nucleon-nucleon interaction (ref. 1). An evident reason for this is that it constitutes one state of the nucleon-nucleon system only. The way in which nuclei are of significance to the nuclear force problem is that empirical data from the nucleon-nucleon scattering experiments can be used to make predictions about observables of the nuclei. Due to the great technical problems in handling the many-body system, the empirical data concerning nuclei are not very relevant to the understanding of the forces between two nucleons. For these reasons we shall confine our discussion of the analysis of the empirical data to the two-nucleon scattering experimental data only.

The empirical data of these scattering experiments consists of the intensity of the scattered beam as a function of the scattering angle. Additional information can be obtained if use is made of polarized beams.

The polarization can be described as the average spin-state of the particles in a beam. The most general scattering problem (ref. 2) is that of finding the polarization and the intensity of the outgoing beam as a function of the scattering angle, for a specified polarization of the incident beam.

In section two we discuss the general formalism that is used in handling the data from polarized beam experiments. The analysis of the data is greatly facilitated if we assume a number of conservation laws to hold. In section three we briefly consider the various conservation laws. The conservation laws allow us to express the data in a convenient way in terms of a limited set of parameters : the Wolfenstein parameters, which are a function of the energy of the beam and the scattering angle. The Wolfenstein parameters are also discussed in section three. Another possibility is to describe the experimental data by a set of phase parameters. This is done in phase-shift analyses of the data. The convenience of phase-shift analyses is that theoretical predictions are usually given in the form of a set of phase parameters. The phase parameters and their relation to the Wolfenstein parameters is discussed in section four.

A third possibility to represent the data is to use a potential. Non-relativistically, one can solve the two-body problem using the Schrödinger equation in which the interaction is expressed in the form of a potential. This concept is useful in the non-relativistic regime below the particle production threshold.

We can distinguish between two aims in the potential description. One is to represent the experimental data as accurately as possible and the second is that it should be convenient to use in many-nucleon problems.

It is not easy to combine both aims in one potential. In agreement with the earlier remarks we shall be interested in potentials which are primarily constructed to represent the nucleon-nucleon scattering data correctly. The potential can be considered as an alternative way of describing the experimental data. It is regarded as a virtue of a potential if the fit can be made with relatively few free parameters. Besides such phenomenological potentials, one can design potentials using meson field theoretical arguments. These are generally referred to as Boson exchange potentials. We shall defer the discussion of these Boson exchange potentials to the chapters three and four. In this chapter we shall confine ourselves mainly to the phenomenological potentials.

In section five we show how the phase parameters can be derived from a potential. A general discussion about the form of the nuclear potential is given in section six. A brief review of a number of phenomenological potentials is given in section seven.

A problem of the potential approach is that a potential is an ill-defined concept in a relativistic description. Relativistic effects may be incorporated to some extent by using non-static, or equivalently, non-local potentials. In a fully relativistic theory however, there is no place for potentials. A discussion on the importance of relativistic effects and the applicability of potentials for elastic nucleon-nucleon scattering is given in section eight.

2) The Formal Analysis of Polarization Experiments

The state of a nucleon is determined completely, if we know its momentum and spin state (for simplicity we leave the iso-spin out of the consideration in this section). One can therefore describe the scattering of two nucleons by using the transition matrix $M(\vec{k}', \vec{k})_{s \rightarrow s'}$. In this expression, the spin state of the two particles is denoted by s and s' , respectively for the initial and final states. \vec{k} and \vec{k}' denote the relative momenta of the two nucleons, respectively before and after the interaction. The matrix elements are the scattering amplitudes of the final states $|\vec{k}', s'\rangle$ for a given initial state $|\vec{k}, s\rangle$.

In scattering experiments usually the momentum can be considered as well-defined. The nucleons in a polarized beam or a polarized target are in general not in the same spin states, so one is dealing with mixtures of pure spin states. In the formalism that is used for dealing with such situations, the mixed state is represented in terms of the so called density operator, introduced by von Neuman (ref. 5).

In this section we indicate how the observables are related to the transition matrix. First we consider the density operator, which is used to describe the spin state of the mixture. A density operator ρ can in general be written as :

$$\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i| ; \quad \sum_i p_i = 1 \quad (1)$$

Where $|\psi_i\rangle$ is a pure state and p_i is the fraction of this pure state in the mixture. The average of the expectation value of a hermitian operator A for this

mixture is :

$$\langle A \rangle = \sum_i p_i \langle \Psi_i | A | \Psi_i \rangle ,$$

which can be written as :

$$= S_\lambda \sum_i p_i \langle \Psi_i | \lambda \rangle \langle \lambda | A | \Psi_i \rangle \quad (2)$$

$$= S_\lambda \sum_i p_i \langle \lambda | A | \Psi_i \rangle \langle \Psi_i | \lambda \rangle = \text{Tr}(A \rho)$$

In (2) $\{|\lambda\rangle\}$ is a complete set of states and $\text{Tr}(\dots)$ denotes the trace.

We now consider a situation in which all particles are in a pure eigenstate of the momentum operator. The mixture in this case is composed of the different spin states. We can write, therefore, the density operator as :

$$\begin{aligned} \rho &= |\vec{k}\rangle \langle \vec{k}| \sum_s p_s(\vec{k}) |s\rangle \langle s| \\ &= |\vec{k}\rangle \langle \vec{k}| \rho(\vec{k}) \quad ; \quad \sum_s p_s(\vec{k}) = 1 \end{aligned} \quad (3)$$

In (3) $|s\rangle$ denotes a function of the spin variables of the two nucleons and $|\vec{k}\rangle$ an eigenstate of the relative momentum operator. As the density operator is hermitian, we can always take for the complete set of spin functions $\{|s\rangle\}$ an orthonormal one. The fraction $p_s(\vec{k})$ for the various pure spin states may in general be dependent on \vec{k} . We notice that $\rho(\vec{k})$ is an operator in spin-space only.

We can use the well-known theorem, which says that on an N -dimensional Hilbert space there act N^2 independent hermitian operators. For the case of the combined spin-space of the nucleons, we can therefore write each spin operator as a linear combination of the sixteen operators $\{\sigma_\mu^1 \sigma_\nu^2\}$, in which $\{\sigma_\mu^1\}$ corresponds to a Pauli

matrix or the identity matrix. The labels ¹ and ² identify the nucleons. For those operators the orthogonality relation :

$$1/4 \text{ Tr } \sigma_{\mu}^1 \sigma_{\nu}^2 \sigma_{\lambda}^1 \sigma_{\rho}^2 = \delta_{\mu\lambda} \delta_{\nu\rho} \quad \text{holds.} \quad (4)$$

Therefore $\rho(\vec{k})$ can be expanded in $\{\sigma_{\mu}^1 \sigma_{\nu}^2\}$ as :

$$\rho(\vec{k}) = 1/4 \sum_{\mu\nu} \text{Tr}\{\sigma_{\mu}^1 \sigma_{\nu}^2 \rho(\vec{k})\} \cdot \sigma_{\mu}^1 \sigma_{\nu}^2 \quad (5)$$

The expectation value of a spin-space operator A_s for a mixture described by $\rho(\vec{k})$ is :

$$\langle A_s \rangle_{\vec{k}} = \text{Tr}\{A_s \rho(\vec{k})\} \quad (6)$$

$$\text{so : } \langle \sigma_{\mu}^1 \sigma_{\nu}^2 \rangle_{\vec{k}} = \text{Tr}\{\sigma_{\mu}^1 \sigma_{\nu}^2 \rho(\vec{k})\} \quad (7)$$

Combining (5) and (7) :

$$\rho(\vec{k}) = 1/4 \sum_{\mu\nu} \langle \sigma_{\mu}^1 \sigma_{\nu}^2 \rangle_{\vec{k}} \cdot \sigma_{\mu}^1 \sigma_{\nu}^2 \quad (8)$$

From (8) we see that the spin state density operator $\rho(\vec{k})$ is determined completely by the sixteen quantities $\langle \sigma_{\mu}^1 \sigma_{\nu}^2 \rangle_{\vec{k}}$.

As the relative momentum of the nucleons is well-defined before the interaction takes place, we can describe the initial state of the nucleons by a density operator ρ_1 of the form 3.

The density operator for the scattered particles is defined by :

$$\rho_{sc} = M \rho_1 M^{\dagger} \quad M \text{ is the transition operator} \quad (9)$$

By substitution of (3) in (9) we see that, using

$$M(\vec{k}', \vec{k}) \equiv \langle \vec{k}' | M | \vec{k} \rangle :$$

$$\rho_{SC}(\vec{k}') \equiv \langle \vec{k}' | \rho_{SC} | \vec{k}' \rangle = M(\vec{k}, \vec{k}') \rho_i(\vec{k}) M^\dagger(\vec{k}, \vec{k}') \quad (10)$$

Taking the trace of $\rho_{SC}(\vec{k}')$ with respect to the complete set of spin states $\{|s\rangle\}$, we get :

$$\text{Tr}\{\rho_{SC}(\vec{k}')\} = \sum_{s, s'} |M(\vec{k}, \vec{k}')|_{s \rightarrow s'}^2 p_{i,s}(\vec{k}) \quad (11)$$

where $p_{i,s}(\vec{k})$ is defined for $\rho_i(\vec{k})$.

From the definition of $\{M(\vec{k}', \vec{k})\}_{s \rightarrow s'}$ as a set of scattering amplitudes, we can interpret $\text{Tr}\{\rho_{SC}(\vec{k}')\}$ as a differential cross-section :

$$\frac{d\sigma}{d\Omega} = I(\vec{k}', \vec{k}) = \text{Tr}\{\rho_{SC}(\vec{k}')\} \quad (12)$$

The differential cross-section $I(\vec{k}', \vec{k})$ is a measurable quantity, as are in principal the correlation functions $\langle \sigma_\mu^1 \sigma_\nu^2 \rangle_{\vec{k}}$ (if σ_μ^1 or σ_ν^2 is the identity matrix, we have in fact a polarization component). We can derive a useful relation between these observables and the transition matrix elements. We can write :

$$\begin{aligned} \rho_{SC}(\vec{k}') &= \langle \vec{k}' | M | \vec{k} \rangle \{1/4 \sum_{\mu\nu} \langle \sigma_\mu^1 \sigma_\nu^2 \rangle_{\vec{k}} \cdot \sigma_\mu^1 \sigma_\nu^2\} \langle \vec{k} | M^\dagger | \vec{k}' \rangle \\ &= 1/4 \sum_{\mu\nu} \langle \sigma_\mu^1 \sigma_\nu^2 \rangle_{\vec{k}} \cdot \sum_{\lambda\rho} Z(\vec{k}', \vec{k})_{\mu\nu, \lambda\rho} \sigma_\lambda^1 \sigma_\rho^2 \end{aligned} \quad (13)$$

where :

$$Z(\vec{k}', \vec{k})_{\mu\nu, \lambda\rho} = 1/4 \text{Tr}\{\sigma_\lambda^1 \sigma_\rho^2 \langle \vec{k}' | M | \vec{k} \rangle \sigma_\mu^1 \sigma_\nu^2 \langle \vec{k} | M^\dagger | \vec{k}' \rangle\} ;$$

$$\langle \sigma_\lambda^1 \sigma_\rho^2 \rangle_{\vec{k}} = \frac{\text{Tr}\{\sigma_\lambda^1 \sigma_\rho^2 \rho_{SC}(\vec{k}')\}}{\text{Tr}\{\rho_{SC}(\vec{k}')\}}, \quad \mu\nu \rightarrow \lambda\rho \quad (14)$$

which is a generalization of (6).

And since there is no correlation between the spins in the initial state, we can write :

$$\langle \sigma_{\mu}^1 \sigma_{\nu}^2 \rangle_{\vec{k}} = \langle \sigma_{\mu}^1 \rangle_{\vec{k}} \langle \sigma_{\nu}^2 \rangle_{\vec{k}} \quad (15)$$

Combining (5), (12), (13), (14) and (15) we get :

*)

$$I(\vec{k}', \vec{k}) \langle \sigma_{\lambda}^1 \sigma_{\rho}^2 \rangle_{\vec{k}} = \sum_{\mu\nu} \langle \sigma_{\mu}^1 \rangle_{\vec{k}} \langle \sigma_{\nu}^2 \rangle_{\vec{k}} \cdot Z(\vec{k}', \vec{k})_{\mu\nu, \lambda\rho} \quad (16)$$

We notice that there are 256 elements $Z(\vec{k}', \vec{k})_{\mu\nu, \lambda\rho}$ and only 16 relations. It should be remembered however that there are 16 complex spinspace elements of $M(\vec{k}', \vec{k})$, therefore at most 32 real numbers are needed to specify these elements completely. From this consideration it is clear that the 256 elements are not independent. Furthermore we have not yet exploited the restrictions due to the various conservation laws.

If the initial polarizations $\langle \vec{\sigma}^i \rangle_{\vec{k}}$ are fixed and we know the elements $Z(\vec{k}', \vec{k})_{\mu\nu, \lambda\rho}$ then we can calculate the lefthand-side of (16). One can measure $I(\vec{k}', \vec{k})$ and at least in principle $\{\langle \sigma_{\lambda}^1 \sigma_{\rho}^2 \rangle_{\vec{k}}\}$. Notice that we cannot directly calculate the elements from the correlation functions and the differential cross-sections.

A detailed discussion on the experimental determination of polarizations and correlation functions is given in reference 2.

*) In reference 2 the summation sign is missing.

3) Invariances and Wolfenstein Parameters

Without taking the various invariances into consideration, one needs at most 32 real numbers to determine the spin-space matrix of M . Due to the invariances, this number can be reduced to 6, if charge independence is not taken into account. Assuming charge independence, one gets the number down to 5. We shall first briefly discuss these invariances and then we show how this reduction comes about.

At present no violation of the conservation laws for energy, momentum and angular momentum are known. So these conservation laws can certainly be used. We shall restrict ourselves to energies in which inelastic effects due to meson production can be ignored. The lightest meson, the pion, has a mass of 140 MeV. This implies that we shall be interested in laboratory kinetic energies up to about 350 MeV. Inelasticities due to Bremsstrahlung can occur, but we shall confine our attention to elastic processes only. For elastic processes we can write :

$$|\vec{k}'| = |\vec{k}| \equiv k \quad (17)$$

As is well-known, the situation with respect to the conservation of parity is different, because the weak interactions are not invariant under space-inversion. Indeed the existence of parity non-conserving components in the nucleon-nucleon interaction seems to be well-established by now. Evidence comes from observations other than elastic nucleon-nucleon scattering experiments (ref. 6,7). The weak interaction is about one million times weaker than the strong interaction, which preserves parity. One expects therefore that the effects of the parity non-conservation are negligible

in nucleon-nucleon scattering. Indeed as reported by Gücker and Thorndike (ref. 8), the ratio of parity non-conservation amplitudes to the amplitudes from the parity conservation part of the interaction is $\approx 0,005$. The level of experimental accuracy is only of the order of 1% of the observables. Therefore, in the analysis of free nucleon-nucleon scattering experiments, one can consider parity to be a conserved quantity.

Time-reversal (T) is related to the parity operation (P) and charge conjugation (C) through the CPT-theorem (ref. 9), which follows from local field theory. The combined CPT-invariance for strong interactions is firmly established. Experiments indicate that the strength of a possible CPT-invariance violating part in the strong-interaction Hamiltonian can be at most 10^{-15} of the CPT-invariant part. Violations of the combined CP-invariance were discovered in 1964 in the decay of the neutral K-meson. It was pointed out that the observed CP-invariance violation and the CPT-theorem can be accounted for, if one assumes the existence of a T-invariance violating part in the strong-interaction Hamiltonian. However experiments indicate that a possible T-invariance violation in strong interactions is too small to this end. No empirical evidence exists for T-invariance violation in strong interactions.

A discussion on P, T, and C-invariances can be found in the references 6 - 11.

The last invariance of interest is charge independence. Charge independence states that the strong interaction is invariant under rotations in iso-spin-space. Crudely speaking, the charge independence hypothesis assumes that the strong interaction does not depend on the charge of the nucleons. A problem with charge independence is that it is very difficult to make

a clear distinction between interactions due to electromagnetic effects and the proper strong interaction (ref. 1, 12). The most trivial of these electro-magnetic effects is the Coulomb force which acts between p-p pairs but not between n-n and p-n pairs. Other electro-magnetic effects are vacuum polarization, (the by the proton charge induced virtual electron-positron pair creation (ref. 13)), magnetic forces due to the magnetic moments of the nucleons and effects due to mass differences between proton and neutron. Furthermore there are effects that become clear in a meson theoretical description, such as mass differences between differently charged mesons. Due to these effects it is not expected that the interaction between n-n, p-n and p-p pairs, after one has corrected for the Coulomb force, is exactly the same. However estimates of the effects mentioned show that these are relatively small, compared to experimental error (ref. 1). It should be mentioned that the data in very low energy scattering experiments (below 10 MeV. lab. kin. energy) are more accurate and it is in this energy range that the assumed charge independence usually is tested. Indeed observations of violations in these experiments have been claimed (ref. 12). Therefore with a possible exception of the analysis of the low energy data, one is justified in using the charge independence hypothesis (ref. 14, 15).

Using the invariance conditions under space-rotations, space-reflections and time-reversal we can determine $M(\vec{k}', \vec{k})$ by a set of 6 so called Wolfenstein parameters. If we furthermore assume charge independence, we can parametrize $M(\vec{k}', \vec{k})$ by 5 Wolfenstein parameters. These parameters determine $M(\vec{k}', \vec{k})$ completely as a function of the energy of the relative

motion ($\sim k^2$) and the scattering angle ($\sim \vec{k}', \vec{k}$). The result is obtained by constructing the most general form of $M(\vec{k}', \vec{k})$ which satisfies the required invariance conditions.

First we construct the most general form of $M(\vec{k}', \vec{k})$, which transforms as a scalar under space-rotations and reflections. To achieve this, we combine the 16 two-nucleon spinspace elements $\sigma_\mu^1 \sigma_\nu^2$ into all possible combinations with well-defined transformation behaviour under space-rotation and inversion. These are listed in table 1.

table 1		
combinations of	$\sigma_\mu^1 \sigma_\nu^2$	transformation behaviour
1		scalar
$\vec{\sigma}^1 \cdot \vec{\sigma}^2 = 1$		scalar
$\vec{\sigma}^1 + \vec{\sigma}^2$		pseudovector
$\vec{\sigma}^1 - \vec{\sigma}^2$		pseudovector
$\vec{\sigma}^1 \times \vec{\sigma}^2$		pseudovector
$t_{\mu\nu} = \sigma_\mu^1 \sigma_\nu^2 + \sigma_\nu^2 \sigma_\mu^1$		symmetric tensor

These spin-space operators should be combined with all possible combinations of the momenta components with a similar transformation behaviour; these are listed in table 2.

table 2	
combinations of $\{k'_\mu, k_\nu\}$	transformation behaviour
1	scalar
$\vec{n} \equiv \vec{k}' \times \vec{k}$	pseudovector
$n_\mu \ n_\nu$	symmetric tensor
$K_\mu \ K_\nu$	symmetric tensor
$P_\mu \ P_\nu$	symmetric tensor
$K_\mu \ P_\nu + P_\nu \ K_\mu$	symmetric tensor

where : $\vec{K} \equiv \vec{k} - \vec{k}'$; $\vec{P} \equiv \vec{k}' + \vec{k}$. These forms may be multiplied by arbitrary functions of the scalars $\vec{k}' \cdot \vec{k}$ and k^2 .

By multiplying forms with a similar transformation behaviour of table 1 and table 2, we find the resulting scalar forms :

table 3 (scalars)		
$1 ; (\vec{\sigma}^1 \cdot \vec{\sigma}^2 - 1)$	$\Sigma_{\mu\nu} \ t_{\mu\nu} \ (n_\mu \ n_\nu)$	$(\vec{\sigma}^1 \cdot \vec{K}) (\vec{\sigma}^2 \cdot \vec{K})$
$(\vec{\sigma}^1 + \vec{\sigma}^2) \cdot \vec{n}$	$\Sigma_{\mu\nu} \ t_{\mu\nu} \ (K_\mu \ K_\nu)$	$(\vec{\sigma}^1 \cdot \vec{P}) (\vec{\sigma}^2 \cdot \vec{P})$
$(\vec{\sigma}^1 - \vec{\sigma}^2) \cdot \vec{n}$	$\Sigma_{\mu\nu} \ t_{\mu\nu} \ (P_\mu \ P_\nu)$	$\{ (\vec{\sigma}^1 \cdot \vec{P}) (\vec{\sigma}^2 \cdot \vec{K}) +$
$(\vec{\sigma}^1 \times \vec{\sigma}^2) \cdot \vec{n}$	$\Sigma_{\mu\nu} \ t_{\mu\nu} \ (K_\mu \ P_\nu + P_\nu \ K_\mu)$	$(\vec{\sigma}^1 \cdot \vec{K}) (\vec{\sigma}^2 \cdot \vec{P}) \}$

Not all these scalar combinations are independent. The symmetric tensor combinations (middle column) are equivalent to the three combinations in the right column.

Next we examine which of these forms is also invariant under time-inversion. From :

$$\begin{aligned}
 \tau^{-1} \vec{\sigma}^i \tau &= -\vec{\sigma}^i & \tau^{-1} \vec{n} \tau &= -\vec{n} & \tau &: \text{time-inversion-operator} \\
 \tau^{-1} \vec{K} \tau &= \vec{K} & \tau^{-1} \vec{P} \tau &= -\vec{P}
 \end{aligned}
 \tag{18}$$

we see that :

$$\tau^{-1} (\vec{\sigma}^1 \times \vec{\sigma}^2) \cdot \vec{n} \tau = -(\vec{\sigma}^1 \times \vec{\sigma}^2) \cdot \vec{n} \quad (19a)$$

$$\tau^{-1} \{ (\vec{\sigma}^1 \cdot \vec{P}) (\vec{\sigma}^2 \cdot \vec{K}) + (\vec{\sigma}^2 \cdot \vec{K}) (\vec{\sigma}^1 \cdot \vec{P}) \} \tau = -\{ (\vec{\sigma}^1 \cdot \vec{P}) (\vec{\sigma}^2 \cdot \vec{K}) + (\vec{\sigma}^2 \cdot \vec{K}) (\vec{\sigma}^1 \cdot \vec{P}) \} \quad (19b)$$

Therefore these forms are ruled out.

If we only consider p-p or n-n scattering then because the particles are identical, $\langle \vec{k}' | M | \vec{k} \rangle$ should be invariant under the exchange of the particle labels 1 and 2. This rules out $(\vec{\sigma}^1 - \vec{\sigma}^2) \cdot \vec{n}$. If we also impose the condition of charge invariance then this term is also ruled out for the case of p-n scattering.

From these considerations we find the most general form of $M(\vec{k}', \vec{k})$, which meets the invariance requirements :

$$M(\vec{k}', \vec{k}) = A + B (\vec{\sigma}^1 \cdot \vec{\sigma}^2 - 1) + C (\vec{\sigma}^1 + \vec{\sigma}^2) \cdot \vec{n} + D (\vec{\sigma}^1 \cdot \vec{K}) (\vec{\sigma}^2 \cdot \vec{K}) + E (\vec{\sigma}^1 \cdot \vec{P}) (\vec{\sigma}^2 \cdot \vec{P}) \quad (20)$$

In this form, the Wolfenstein parameters A, B, C, D and E are in general functions of k^2 and \vec{k}', \vec{k} .

It is possible to express the coefficients $Z(\vec{k}', \vec{k})_{\mu\nu, \lambda\rho}$ in terms of these 5 Wolfenstein parameters.

Complete lists of these expressions can, for instance, be found in the review article by Macgregor, Moravcsik and Stapp (ref. 2). Furthermore it should be mentioned that other ways of parametrizing $M(\vec{k}', \vec{k})$ then according to (20), are used.

4) The Phase Parameters

Theoretical predictions about the results of scattering experiments are usually given in the form of a set of phase parameters. We have indicated how the observables can be used to determine the transition matrix $M(\vec{k}', \vec{k})_{s \rightarrow s'}$. In this section we show how the set of phase parameters can be extracted from the transition matrix. The theoretical phase parameters can be compared to these experimental ones.

The phase parameters are related to the T-matrix, so one needs a relation between the T-matrix and the M-matrix. We shall first derive this relation and then we show what the relation is between the T-matrix and the phase parameters. We begin by considering the situation without the Coulomb field; after the derivations we indicate what modifications are necessary to include a Coulomb field, for the case of p-p scattering.

The solution of a stationary scattering problem for large distances between the particles can be written as :

$$\Psi \xrightarrow{r \rightarrow \infty} \Psi_{in} + \Psi_{sc} \quad (21)$$

Ψ_{in} and Ψ_{sc} represent respectively the incoming and the scattered waves. We shall in this discussion confine ourselves to the simplifying case in which the beams consist of particles in a pure spin state.

We use the singlet-triplet representation $|S, m_s\rangle$ and approximate the incoming beam by a plane wave which travels in the direction of the positive z-axis:

$$\Psi_{in} = e^{ikz} |S, m_s\rangle \quad (22a)$$

$$= (4\pi) \sum_L \left[\frac{(2L+1)}{4\pi} \right]^{1/2} [i^L j_L(kr)] Y_{L,0}(\Omega) |S, m_s\rangle \quad (22b)$$

In (22b) we have applied the Bauer expansion of the plane wave (ref. 16), $j_L(kr)$ is a spherical Bessel function, and $Y(\Omega)$ is a spherical harmonic function.

$L, 0$

For large values of r we get :

$$\psi_{in} \xrightarrow{r \rightarrow \infty} \frac{4\pi}{2ikr} (e^{ikr} - e^{-ikr}) \sum_L \left[\frac{2L+1}{4\pi} \right]^{1/2} \cdot Y(\Omega) \begin{vmatrix} Sm_S \\ L, 0 \end{vmatrix} \quad (23)$$

We shall use the notation :

$$\psi_{in} \xrightarrow{r \rightarrow \infty} \frac{1}{r} (e^{ikr} - e^{-ikr}) \begin{vmatrix} \Omega, \sigma^1, \sigma^2 \\ k, in \end{vmatrix} \quad (24)$$

The scattered wave for the incoming beam (22b) can be written as :

$$\begin{aligned} \psi_{sc} \xrightarrow{r \rightarrow \infty} \frac{e^{ikr}}{r} \sum_{S'm'_S} M(\vec{k}', \vec{k})_{Sm_S \rightarrow S'm'_S} \begin{vmatrix} S'm'_S \end{vmatrix} \\ = \frac{e^{ikr}}{r} \begin{vmatrix} \Omega, \sigma^1, \sigma^2 \\ k, sc \end{vmatrix} \end{aligned} \quad (25)$$

Ω denotes the direction of \vec{k}' with respect to the z -axis. The T-matrix can be introduced by :

$$\begin{vmatrix} \Omega, \sigma^1, \sigma^2 \\ k, sc \end{vmatrix} = T \begin{vmatrix} \Omega, \sigma^1, \sigma^2 \\ k, in \end{vmatrix} \quad (26)$$

From the formulas (23)-(26) we can conclude :

$$\sum_{S'm'_S} M(\vec{k}', \vec{k})_{Sm_S \rightarrow S'm'_S} \begin{vmatrix} S'm'_S \end{vmatrix} = \frac{4\pi}{2ik} \sum_L \left[\frac{2L+1}{4\pi} \right]^{1/2} \cdot T \begin{vmatrix} Y(\Omega) \\ L, 0 \end{vmatrix} \begin{vmatrix} Sm_S \end{vmatrix} \quad (27)$$

$M(\vec{k}', \vec{k})_{Sm_S \rightarrow S'm'_S}$ can be expanded in spherical harmonics :

$$M(\vec{k}', \vec{k})_{S m_S \rightarrow S' m'_S} = \sum_{L' m'_L} M(L' m'_L S' m'_S, S m_S) Y(\Omega)_{L' m'_L} \quad (28)$$

From the completeness of the set $\{Y(\Omega) | S m_S \rangle\}_{L m_L}$ we find the relation :

$$M(L' m'_L S' m'_S, S m_S)_k = \frac{\sqrt{\pi}}{ik} \sum_L (2L+1)^{1/2} \langle L' m'_L S' m'_S | T | L 0 S m_S \rangle \quad (29)$$

where :

$$\langle L' m'_L S' m'_S | T | L 0 S m_S \rangle = \int d\Omega Y_{L' m'_L}^*(\Omega) \langle S' m'_S | T | S m_S \rangle Y_{L 0}(\Omega) \quad (30)$$

Because of the conservation of parity, charge independence and the Pauli principle, the magnitude of the spin is conserved. In order to exhibit the consequence of the rotational invariance more clearly, we transform from the $\{L m_L S m_S\}$ -representation to the $\{J M L S\}$ -representation, in which J is the total angular momentum of the system and M its z-component :

$$|L m_L S m_S \rangle = \sum_{J M} (L S m_L m_S | J M) |J M L S \rangle \quad (31)$$

where $(L S m_L m_S | J M)$ is a Clebsch-Gordan coefficient.

The matrix element $\langle J' M' L' S' | T | J M L S \rangle$ is diagonal in J and independent of M due to the rotational invariance. So we see that :

$$M(L' m'_L S m_S; S m'_S)_k = \frac{\sqrt{\pi}}{ik} \sum_{L J} (2L+1)^{1/2} \langle J M L' S | T | J M L S \rangle \cdot \sum_M (L' S m'_L m'_S | J M) (L S 0 m_S | J M) \quad (32)$$

S takes only the values 0 and 1, therefore the only non-vanishing matrix elements are :

(33)

$$\text{singlet case : } S=0 \quad T_J = \langle JM \ L=J \ 0 \ | T | \ JM \ L=J \ 0 \rangle$$

$$\text{triplet case : } S=1 \quad T_{J,J} = \langle JM \ L=J \ 1 \ | T | \ JM \ L=J \ 1 \rangle$$

$$T_{J\pm 1,J} = \langle JM \ L=J\pm 1 \ 1 \ | T | \ JM \ L=J\pm 1 \ 1 \rangle$$

$$T^J = \langle JM \ L=J\pm 1 \ 1 \ | T | \ JM \ L=J\mp 1 \ 1 \rangle$$

(34)

In these expressions we used the notation of Stapp, Ypsilantis and Metropolis (ref. 17). Time-reversal invariance requires the T-matrix to be symmetric, so the two elements T^J are equal. The S-matrix, which is related to the T-matrix by $S_{ij} = T_{ij} + \delta_{ij}$ should be unitary. This is a consequence of the required conservation of probability. The unitarity requirement leads to a description of the T-matrix in terms of phase-parameters.

The unitarity of the S-matrix implies that the T-matrix elements T_J and $T_{J,J}$ can be written as :

$$\text{for the singlet case : } T_J = e^{2i\delta_J^1} - 1 \quad (35a)$$

$$\text{for the triplet case : } T_{J,J} = e^{2i\delta_J^3} - 1 \quad (35b)$$

where δ_J^1 and δ_J^3 are real.

In the spin-triplet case the two states for $L=J+1$ and $L=J-1$ are mixed. Due to this mixing we can not describe the scattering in terms of phase-shifts only. A parametrization which meets both the unitarity and the symmetry requirements is :

$$T_{J\pm 1,J} = \cos 2\bar{\epsilon}_J e^{2i\bar{\delta}_{J\pm 1,J}} - 1 \quad (36a)$$

$$T^J = i \sin 2\bar{\epsilon}_J e^{i(\bar{\delta}_{J+1,J} + \bar{\delta}_{J-1,J})} \quad (36b)$$

In these expressions $\bar{\epsilon}_J$ is the mixing parameter, which is a measure of the extent to which the $L=J+1$ and $L=J-1$ states are mixed. $\bar{\delta}_{J+1,J}$ and $\bar{\delta}_{J-1,J}$ are the so called bar phase-shifts (ref. 17). We notice that there are 5 phase parameters for each value of J . The relations between the Wolfenstein parameters and these phase parameters are obtained through equations (20), (29), (32)-(36).

It is convenient for reference in the next sections to develop this formalism a little further. After substitution of (21), (22), (25) and (28) into (32), the solution of the scattering problem for large values of r in terms of the T-matrix elements is found to be :

$$\begin{aligned} \Psi \xrightarrow{r \rightarrow \infty} & \frac{\sqrt{\pi}}{ikr} (e^{ikr} - e^{-ikr}) \sum_{LJM} (2L+1)^{1/2} (L0m_s | JM) | JMLS \rangle \\ & + \frac{\sqrt{\pi}}{ikr} e^{ikr} \sum_{L'L'JM} (2L'+1)^{1/2} (L'0m_s | J'M') \cdot \\ & \cdot \langle JML'S | T | JMLS \rangle | JML'S \rangle \end{aligned} \quad (37)$$

The nuclear force has a finite range R . The wave functions outside this region of interaction can be analyzed in terms of partial waves. We consider the solutions of the radial Schrödinger equation for $r > R$ which have the asymptotic form $\sim \frac{e^{ikr}}{kr}$ and $\sim \frac{e^{-ikr}}{kr}$ respectively. These solutions can be written as $O_L(kr)$ and $I_L(kr)$:

$$O_L(kr) \xrightarrow{r \rightarrow \infty} \frac{e^{ikr}}{kr} ; \quad I_L(kr) \xrightarrow{r \rightarrow \infty} \frac{e^{-ikr}}{kr} \quad (38)$$

$O_L(kr)$ and $I_L(kr)$ are related to spherical Hankel functions $h_L^+(kr)$ and $h_L^-(kr)$ by :

$$O_L(kr) = i^L h_L^+(kr) \quad \text{and} \quad I_L(kr) = (-i)^L h_L^-(kr) \quad (39)$$

Using the relation $T_{ij} = S_{ij} - \delta_{ij}$ we find for the wave function for $r > R$, but not necessarily $r \rightarrow \infty$:

$$\begin{aligned} \Psi_{r>R} = & \frac{\sqrt{\pi}}{i} \sum_{LL',JM} (2L+1)^{1/2} (L O S m_S | J M) \cdot \\ & \cdot \{ -I_L(kr) \delta_{L'L} + O_{L'}(kr) \langle J M L' S | S | J M L S \rangle \} \cdot \\ & \cdot |J M L' S \rangle \end{aligned} \quad (40)$$

$I_L(kr)$ can be interpreted as an incoming spherical wave and $O_{L'}(kr)$ as an outgoing spherical wave. The S-matrix element in the bracket of (40) determines the probability of a resulting outgoing wave $O_{L'}(kr)$ for an incoming wave $I_L(kr)$, for a total angular momentum J , M and a spin magnitude S .

In case of p-p scattering we have to include the Coulomb potential. Usually one makes the approximation that inside the region of the strong nucleon-nucleon interaction, the Coulomb potential can be neglected. In view of the relative strength of the strong interaction, this approximation is reasonable. In this case we can define the nucleon phase parameters in a similar way as without the Coulomb force (ref.2). The difference from the above formalism is that instead of the spherical Hankel functions $h_L^+(kr)$ and $h_L^-(kr)$ we have to use, respectively, the Coulomb functions $u_L^+(kr) e^{i\sigma_L}$ and $u_L^-(kr) e^{-i\sigma_L}$. The Coulomb functions are solutions of the radial Schrödinger equation in which the Coulomb potential is included in the Hamilton operator. $u_L^+(kr)$ and $u_L^-(kr)$ have the asymptotic form :

$$u_L^+(kr) \xrightarrow{r \rightarrow \infty} (-i)^L e^{i(kr - \gamma \ln kr)} \quad (41a)$$

$$u_L^-(kr) \xrightarrow{r \rightarrow \infty} i^L e^{-i(kr - \gamma \ln kr)} \quad (41b)$$

In the notation of Messiah (ref. 16) σ_L is the Coulomb phase-shift, $\gamma = \frac{e^2}{\hbar v}$ and v the incident velocity.

In the preceding discussion we have indicated how the observables are related to the phase parameters. It is however not at all trivial to calculate the phase-shifts directly, once the observables are known (ref. 18). Although one can calculate the observables if we know the phase-shifts. The way in which one proceeds is as follows : One assumes a plausible set of phase-shifts, calculates the observables and compares these with the experimental data. Then one tries to improve the fit by a systematic alternation of the phase-shifts. In this process one aims at minimizing :

$$\chi^2(\delta) = \sum_i^n \left(\frac{\Delta_i}{\epsilon_i} \right)^2$$

(42)

Δ_i : the difference of the experimental and calculated value of the observable i
 ϵ_i : the experimental error
 n : total number of data

The quantity $\chi^2(\delta)$ is generally referred to as the 'goodness of fit' parameter, although Signell (ref. 20) calls it the 'badness of fit' parameter, which may be more appropriate. Phase-shift analyses have been performed by a number of groups. Especially known are those of Arndt and MacGregor from Livermore and those of Breit and his Yale-group. A typical value for the goodness of fit per data point is about 1 for laboratory energies less than 400 MeV. (ref. 1).

5) Phase Parameters and Potentials

In the previous section we saw how one represents the empirical data by a set of phase parameters. Another possibility is to use a phenomenological potential from which phase parameters can be derived, which are in agreement with the experimental data. The degree to which such a phenomenological potential is adequate, is given by the 'goodness of fit' parameter.

The reason one wants to do this can be that theoretical arguments are available to partially justify the use of a given form of the potential. Furthermore a potential can, in conjunction with a Schrödinger equation, be used to calculate properties of many-body systems. If one looks for applications to many-body systems one should keep in mind that a very precise fit to on-shell data in nucleon-nucleon scattering does not necessarily imply that the off-shell data are well-described. Last but not least, a consideration is that a potential is a very familiar concept which can help to get some insight in the properties of the interaction.

In this section we shall show how potentials can be related to the phase parameters as defined in the previous section.

The Schrödinger equation for two nucleons which interact through a potential $V(\vec{r}, \vec{\sigma}^1, \vec{\sigma}^2)$, in which \vec{r} is the relative position of the nucleons, is :

$$\left\{ \Delta - \frac{2m}{\hbar^2} V(\vec{r}, \vec{\sigma}^1, \vec{\sigma}^2) + k^2 \right\} \psi(k, \vec{r}, \vec{\sigma}^1, \vec{\sigma}^2) = 0 \quad (43)$$

We assume that the mass of both nucleons is the same. The set of functions $\{|JMLS\rangle\}$ as defined in (31) is orthonormal and complete. We can expand $\psi(k, \vec{r}, \vec{\sigma}^1, \vec{\sigma}^2)$ in

this set. Either the spin-singlet or the spin-triplet case can be considered, as these do not mix :

$$\psi_S(k, \vec{r}, \vec{\sigma}^1, \vec{\sigma}^2) = \sum_{JML} \frac{1}{r} u(kr) |JMLS\rangle \quad (44)$$

By substitution in the Schrödinger equation we can derive for the radial part :

$$\left\{ \frac{d^2}{dr^2} + k^2 - L'(L'+1)r^{-2} \right\} u(kr) = \sum_{JML'S} \frac{1}{r} \langle JML'S | V(\vec{r}, \vec{\sigma}^1, \vec{\sigma}^2) | JMLS \rangle u(kr) \quad (45)$$

In (45) we employed the rotational invariance of the potential. Without potential, (45) reduces to :

$$\left\{ \frac{d^2}{dr^2} + k^2 - L'(L'+1)r^{-2} \right\} kr i^{L'} j_{L'}(kr) = 0 \quad (46)$$

$j_{L'}(kr)$ is the regular spherical Bessel function. By multiplying (45) on the left hand side by $kr i^{L'} j_{L'}(kr)$ and (46) by $u(kr)$, we get by subtraction of the two equations and an integration over r :

$$\lim_{r \rightarrow \infty} k i^{L'} \int_0^r W\{r j_{L'}(kr), u(kr)\} dr = \sum_{JML'S} \frac{m}{\hbar^2} i^{L'} \int_0^R dr \langle JML'S | V(\vec{r}, \vec{\sigma}^1, \vec{\sigma}^2) | JMLS \rangle u(kr) kr j_{L'}(kr) \quad (47)$$

where $W\{.,.\}$ is the Wronskian. From (40) and (44) we see that :

$$\lim_{r \rightarrow \infty} u(kr) = \frac{\sqrt{\pi}}{i} \sum_L (2L+1)^{1/2} (L0S m_S | J m_S) \cdot \{-e^{-ikr} \delta_{L'L} + e^{ikr} \langle JML'S | S | JMLS \rangle\} \quad (48)$$

After substitution of (48) in (47) and working out the Wronskian, we get :

$$\begin{aligned} \frac{\sqrt{\pi}}{i} \Sigma_L (2L+1)^{1/2} (L0m_S | Jm_S) \{ \delta_{L'L} - \langle Jm_L'S | S | Jm_L S \rangle \} = \\ \Sigma_L \frac{mi^{L'}}{\hbar^2} \int_0^R dr \langle Jm_L'S | V(\vec{r}, \vec{\sigma}^1, \vec{\sigma}^2) | Jm_L S \rangle \frac{u(kr)}{Jm_L S} r j_{L'}(kr) \end{aligned} \quad (49)$$

Using (49) and $T_{ij} = S_{ij} - \delta_{ij}$, we can identify :

$$\begin{aligned} \langle Jm_L'S | T | Jm_L S \rangle = \frac{mi^{L'+1}}{\sqrt{\pi} \hbar^2 (2L+1)^{1/2} (L0m_S | Jm_S)} \cdot \\ \int_0^R dr \langle Jm_L'S | V(\vec{r}, \vec{\sigma}^1, \vec{\sigma}^2) | Jm_L S \rangle \frac{u(kr)}{Jm_L S} r j_{L'}(kr) \end{aligned} \quad (50)$$

So we find the following relations between the phase parameters and the potential :

$$\begin{aligned} e^{2i\delta_J^1} - 1 &= \frac{mi^{J+1}}{\sqrt{\pi} \hbar^2 (2J+1)^{1/2}} \cdot \\ \int_0^R dr \langle JMJO | V(\vec{r}, \vec{\sigma}^1, \vec{\sigma}^2) | JMJO \rangle \frac{u(kr)}{JMJO} r j_J(kr) \end{aligned} \quad (51a)$$

$$\begin{aligned} e^{2i\delta_J^3} - 1 &= \frac{mi^{J+1}}{\sqrt{\pi} \hbar^2 (2J+1)^{1/2} (J0m_S | Jm_S)} \cdot \\ \int_0^R dr \langle JMJO | V(\vec{r}, \vec{\sigma}^1, \vec{\sigma}^2) | JMJO \rangle \frac{u(kr)}{JMJO} r j_J(kr) \end{aligned} \quad (51b)$$

$$\cos 2\epsilon_J e^{2i\delta_{J\pm 1, J}} - 1 = \frac{mi^{J\pm 1+1}}{\sqrt{\pi} \hbar^2 (2J\pm 2+1)^{1/2} (J\pm 1, 0 | m_S | Jm_S)} \cdot$$

$$\begin{aligned} \int_0^R dr \langle JM, J\pm 1, 1 | V(\vec{r}, \vec{\sigma}^1, \vec{\sigma}^2) | JM, J\pm 1, 1 \rangle \frac{u(kr)}{JM, J\pm 1, 1} r j_{J\pm 1}(kr) \end{aligned} \quad (51c)$$

$$i \sin 2\epsilon_J^- e^{i(\bar{\delta}_{J+1,J} + \bar{\delta}_{J-1,J})} = \frac{m i^{J+1+1}}{\sqrt{\pi} \hbar^2 (2J+2+1)^{1/2} (J+1, 0 | m_S | J m_S)}.$$

$$\int_0^R dr \langle JM, J+1, 1 | V(\vec{r}, \vec{\sigma}^1, \vec{\sigma}^2) | JM, J-1, 1 \rangle \frac{u(kr)}{J m_S} \frac{j(kr)}{J+1} \quad (51d)$$

In the first order Born approximation we replace $\frac{u(kr)}{J m_S}$ in the integral by :

$$2\sqrt{\pi} (2L+1)^{1/2} (L 0 | m_S | JM) kr i^L J_L(kr) \quad (52)$$

The factors in (52) are found from the asymptotic behaviours of $\frac{u(r)}{J m_S}$, (eq. 48) and $j_L(kr)$. Using (50) and (52) :

$$\langle JML'S | T | JMLS \rangle_{\text{Born}} = \frac{2mk i^{L+L'+1}}{\hbar^2} \int_0^R dr r^2 \langle JML'S | V(\vec{r}, \vec{\sigma}^1, \vec{\sigma}^2) | JMLS \rangle j_L(kr) j_L(kr) \quad (53)$$

The phase parameter expressions of the T-matrix elements can be expanded. In first order Born approximation, we retain the first order term. We find :

$$\delta_B^1 = - \frac{mk}{\hbar^2} \int_0^R dr r^2 \langle JM J 0 | V(\vec{r}, \vec{\sigma}^1, \vec{\sigma}^2) | JM J 0 \rangle \{j_J(kr)\}^2 \quad (54a)$$

$$\delta_B^3 = - \frac{mk}{\hbar^2} \int_0^R dr r^2 \langle JM J 1 | V(\vec{r}, \vec{\sigma}^1, \vec{\sigma}^2) | JM J 1 \rangle \{j_J(kr)\}^2 \quad (54b)$$

$$\bar{\delta}_{J+1,J}^B = - \frac{mk}{\hbar^2} \int_0^R dr r^2 \langle JM J+1, 1 | V(\vec{r}, \vec{\sigma}^1, \vec{\sigma}^2) | JM J+1, 1 \rangle \frac{\{j(kr)\}^2}{J+1} \quad (54c)$$

$$\epsilon_J^{-B} = - \frac{mk}{\hbar^2} \int_0^R dr r^2 \langle JM J+1, 1 | V(\vec{r}, \vec{\sigma}^1, \vec{\sigma}^2) | JM J-1, 1 \rangle \frac{j(kr)}{J+1} \frac{j(kr)}{J-1} \quad (54d)$$

In reference 19 it is shown that the phase-shifts with orbital angular momentum quantumnumber L ($L \neq 0$) are influenced very little by the behaviour of the potential at distances smaller than one half of the impact parameter b_L , where :

$$b_L = \frac{1}{k} \sqrt{L(L+1)} \quad (55)$$

One can understand this result by considering the form of the spherical Bessel functions which appear in the relations (51). For a few values L these are shown in graph 1.

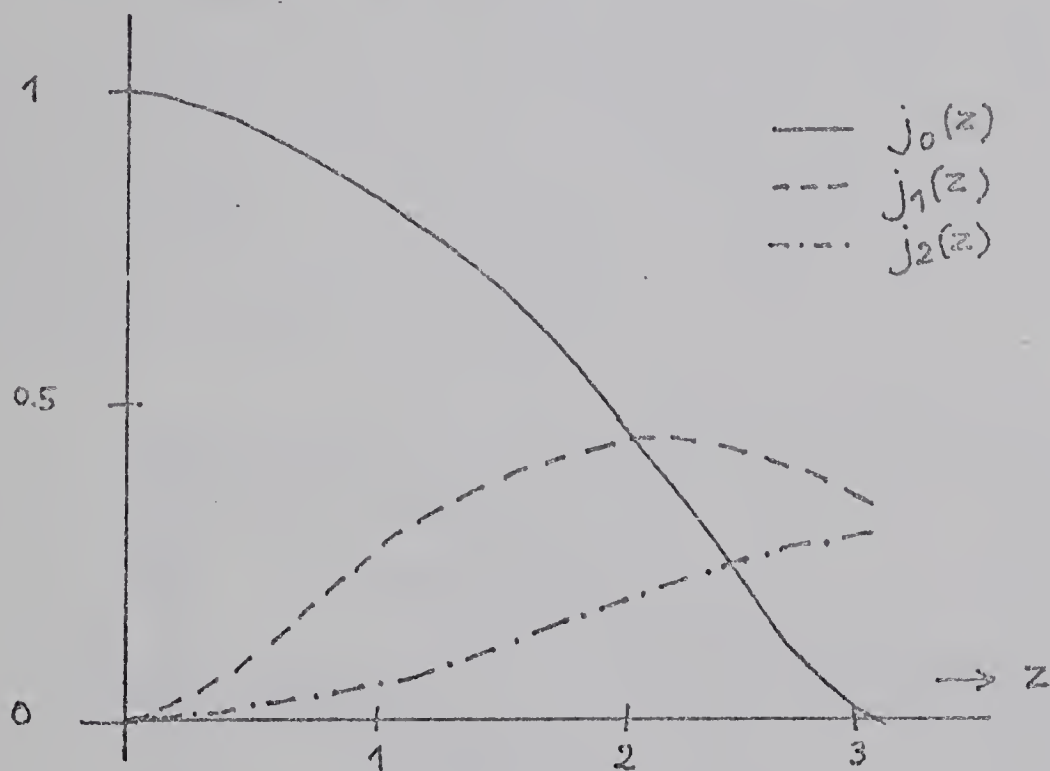


figure 1 :the spherical Bessel functions $j_0(z)$; $j_1(z)$ and $j_2(z)$

Classically the impact parameter represents the nearest approach of the nucleons. In general the Born approximation holds if the strength of the interaction is not too great. The interaction decreases with increasing distance between the nucleons. One may therefore expect

that for not too low values of L and not too high values of k , the Born approximation is justified. From graph 1 one notices the strikingly different behaviour of $j_0(z)$ from the other spherical Bessel functions. Large contributions to the integrals in the relations (51) occur for small values of k . The bound deuteron state appears for $k=0.233 \text{ fm}^{-1}$. The cm-momentum k is related to the laboratory kinetic energy by $E_{\text{lab.}} (\text{MeV.}) = 83 k^2$. Therefore the $k=0.233 \text{ fm}^{-1}$ corresponds to $E_{\text{lab.}} \approx -4.452 \text{ MeV.}$ (ref. 20). The Born approximation can be justified for the S-waves for large values of the energy only.

We denote the first zero in $j_0(z)$ by z_0 . From equations (54) one expects that the S-wave phase-shift will decrease as a function of energy for $k > z_0/R$, where $k_0 = z_0/R$ corresponds to a laboratory energy of about 80 MeV., if we assume $R \approx 3 \text{ fm}$. The S-wave phase-shifts are shown in figure 2.

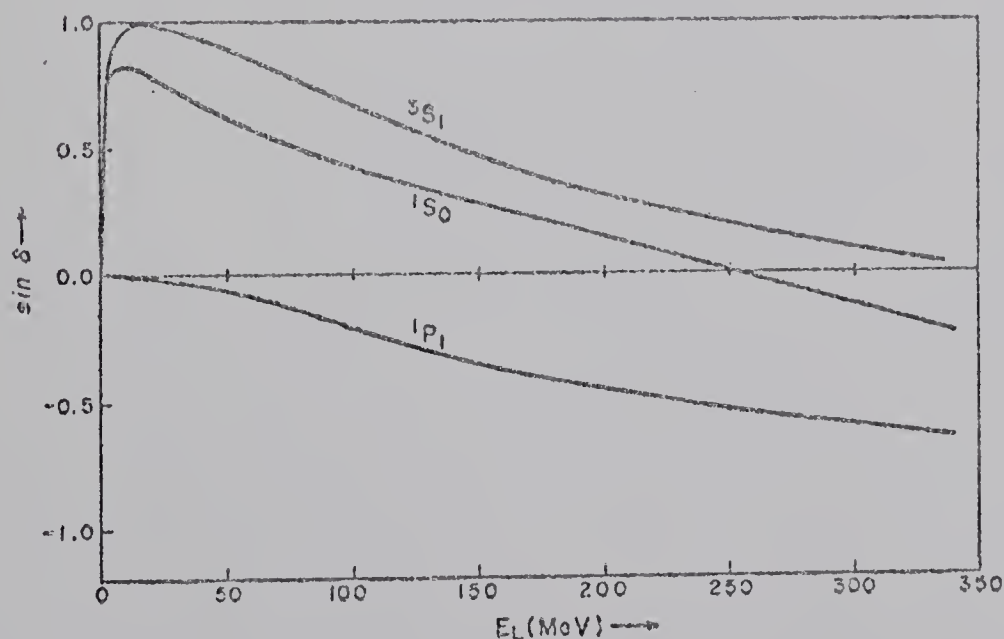


figure 2: the 3S_1 , 1S_0 and 1P_1 phase-shifts (ref.20)

The low energy behaviour of the 3S_1 phase-shift reflects the existence of the deuteron bound state. A detailed discussion on the analysis of the low energy phenomena can be found in reference 4.

Figure 2 also shows the phase-shifts for the singlet P-wave. As can be expected from the behaviour of $j_1(z)$, it increases as a function of the energy. The fact that it is negative, indicates that the potential is repulsive in this state. This can be seen from equations 54. Similarly, an attractive potential corresponds to a positive phase-shift.

6) The General Form of the Potential

The invariances which hold for the strong interactions limit the possible forms of the interaction potential. In this section we shall discuss the general form and consider briefly the behaviour of the various terms in it.

The potential should be a time-reversal invariant scalar. Furthermore we assume charge independence. Due to the identity of the nucleons, the potential should be invariant under the exchange of the particles. Wigner and Eisenbud worked out a general form of the potential that meets these requirements (ref. 22). Furthermore they imposed the ad-hoc restriction that the relative momentum \vec{k} can appear at most linearly in the potential. This condition is of course not necessarily true, but it was hoped to be a reasonable simplification. The condition holds for instance also for atomic electrons. For some time one had hoped that a fit to the data could be made under the even more severe restriction that no momentum dependence is included at all. However one could show that a significant improvement of the fit is possible by the inclusion of a spin-orbit term, in which \vec{k} appears linearly (ref. 23). Especially for energies above 100 MeV. this improvement was considerable.

The general form of the potential can be found in a similar way as in the case of the transition matrix. The potential depends on the separation \vec{r} of the nucleons, their relative momentum \vec{k} , and their spins $\vec{\sigma}^1$ and $\vec{\sigma}^2$. $\vec{L} = \vec{r} \times \vec{k}$ is the only pseudo-vector that can be formed from \vec{r} and \vec{k} . \vec{L} changes sign under time-reversal. Using table 1, we see that under the Wigner-Eisenbud condition the potential should be a combination of :

$$1 \quad ; \quad \vec{\sigma}^1 \cdot \vec{\sigma}^2 \quad ; \quad (\vec{\sigma}^1 \cdot \hat{r})(\vec{\sigma}^2 \cdot \hat{r}) \quad ; \quad (\vec{\sigma}^1 + \vec{\sigma}^2) \cdot (\vec{r} \times \vec{k}) \quad (56)$$

Each of these terms may be multiplied by a function of the magnitude of the separation \vec{r} .

The general form of a scalar in iso-spin space is a combination of the iso-spin space identity and $\vec{\tau}^1, \vec{\tau}^2$. $\vec{\tau}^1$ and $\vec{\tau}^2$ are the iso-spin operators of the nucleons 1 and 2 respectively.

The potential is usually written as :

$$v = v_c(r) + v_t(r) S_{12} + v_{ls}(r) (\vec{\sigma}^1 + \vec{\sigma}^2) \cdot (\vec{r} \times \vec{k}) + v_{ss}(r) (\vec{\sigma}^1 \cdot \vec{\sigma}^2) \quad (57)$$

where : $v_c(r)$: central potential
 $v_t(r)$: tensor potential
 $v_{ls}(r)$: spin-orbit potential
 $v_{ss}(r)$: spin-spin potential

$$S_{12} = \{3 (\vec{\sigma}^1 \cdot \hat{r}) (\vec{\sigma}^2 \cdot \hat{r}) - \vec{\sigma}^1 \cdot \vec{\sigma}^2\} \quad (58)$$

This quantity is called the tensor operator, because $(\vec{\sigma}^1 \cdot \hat{r}) (\vec{\sigma}^2 \cdot \hat{r})$ transforms as a second-rank tensor in coordinate space. The particular combination with $\vec{\sigma}^1 \cdot \vec{\sigma}^2$ is convenient, because by its inclusion, S_{12} disappears for spin-singlet states. This can be seen by using the fact that for the spin-singlet state:

$$\sigma_{\hat{r}}^1 = - \sigma_{\hat{r}}^2 \quad (59)$$

and

$$1/4 (\sigma_1^2 + \sigma_2^2 + 2\vec{\sigma}_1 \cdot \vec{\sigma}_2) = S(S+1) \quad (60)$$

Therefore if $S = 0$, we get : $S_{12} = -3 + 3 = 0$

We notice that the coupling between the two spin-triplet states is entirely due to the tensor potential.

We write each of the functions $v_i(r)$,
 $i \in \{c, t, ls, ss\}$ as :

$$v_i(r) = 1/4 (1 - \vec{\tau}^1 \cdot \vec{\tau}^2) v_i^1(r) + 1/4 (3 + \vec{\tau}^1 \cdot \vec{\tau}^2) v_i^3(r) \quad (61)$$

For the iso-spin singlet case, in which $\vec{\tau}^1 \cdot \vec{\tau}^2 = -3$
 we have $v_i(r) \rightarrow v_i(r) = v_i^1(r)$ and for the iso-spin trip-
 let case $\vec{\tau}^1 \cdot \vec{\tau}^2 = +1$, so then : $v_i(r) \rightarrow v_i(r) = v_i^3(r)$.

A useful rule is that, because of the Pauli
 principle, the sum of the quantum numbers T, S and L
 should be odd. T is the quantum number for the magnitude
 of the total iso-spin. Therefore in the spin-singlet,
 even parity - and spin-triplet, odd parity cases, we are
 dealing with the iso-spin triplet functions $v_i^3(r)$ and
 in both the other cases with the iso-spin singlet func-
 tion $v_i^1(r)$.

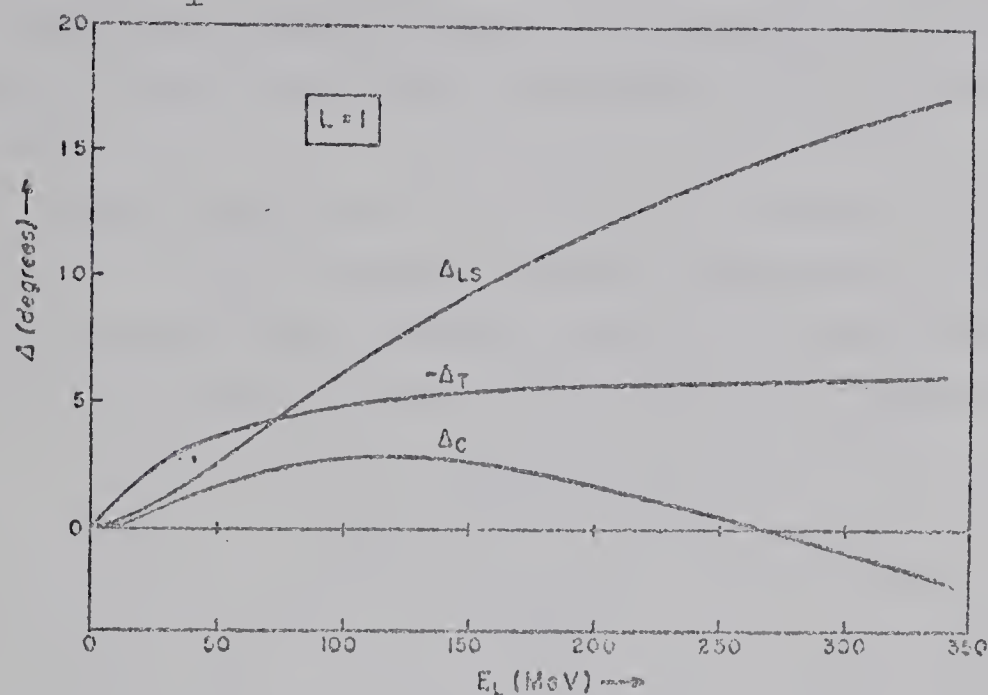
An interesting method of deducing information
 about the functions $v_i(r)$ from the experimental phase-
 shifts is obtained by substitution of (57) in the Born
 approximated phase-shift expressions (54b) and (54c) for
 the spin-triplet case. We get a linear expression of
 these three phase-shifts in terms of the integrals :

$$\Delta_i(k) = \frac{-mk}{\hbar^2} \int_0^R dr r^2 v_i(r) j_J(kr) j_J(kr) \quad (62)$$

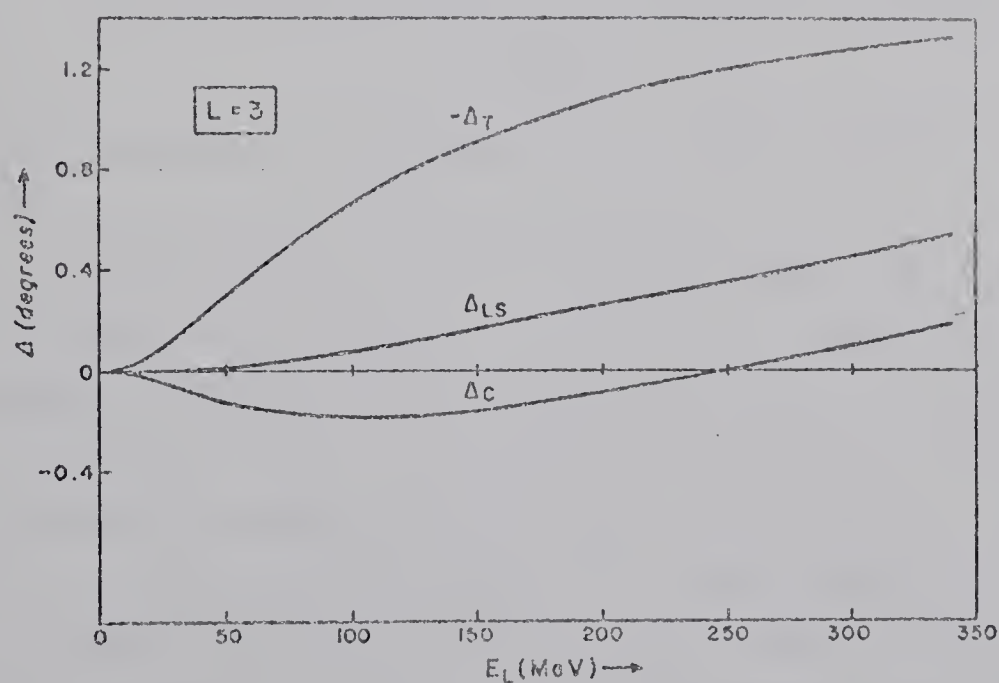
where $v_i(r)$ is $v_c(r) + v_{ss}(r)$, $v_t(r)$ or $v_{ls}(r)$
 respectively. It is therefore possible to express these
 integrals in terms of the phase-shifts.

Of course, this technique can not be applied for
 the S-waves, as the Born approximation does not hold in
 that case. Besides there are only two S-waves, so the
 inversion of the phase-space expressions in terms of
 $\{\Delta_i(k)\}$ can not be made.

In the figures 3, 4 and 5 one can see the results of this technique for $L=1$, $L=2$ and $L=3$ (ref. 20). For $L=1$ and $L=3$ the iso-spin triplet functions v_i^3 appear in the integrals. For $L=2$ the iso-spin singlet functions $v_i^1(r)$.



figure_3: phase-shift components for the $L=1$ -waves (ref.20)

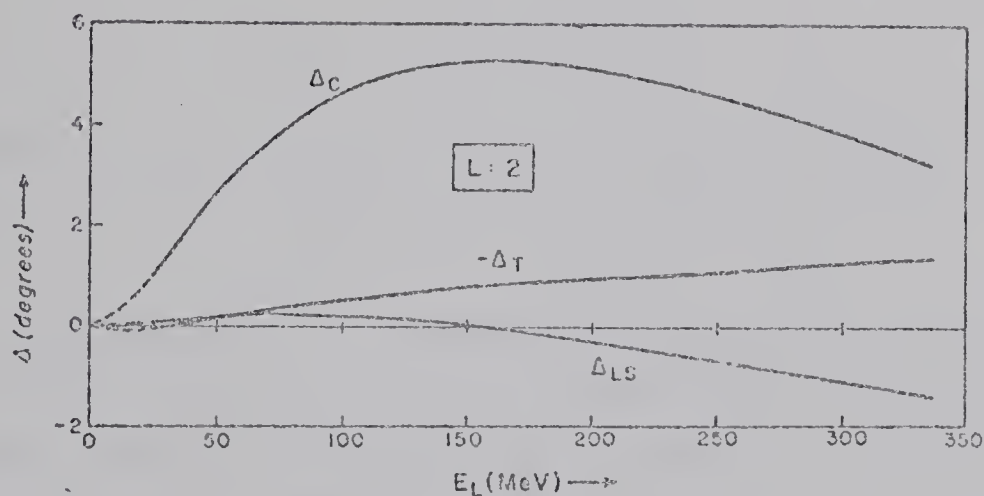


figure_4: phase-shift components for the $L=3$ -waves (ref.20)

In accordance with what was expected, we see that the integrals are considerably larger for P-waves than

for F-waves. In figure 3 we see at shorter distances a dominant attractive spin-orbit potential, a strong repulsive tensor potential and a weakly repulsive central and spin-spin potential. For larger distances the spin-orbit potential becomes weakly attractive, the tensor potential remains strongly repulsive and the central - plus spin-spin potential becomes weakly attractive.

This behaviour at larger distances is confirmed by figure 4. We notice a strong repulsive tensor potential, a relatively weakly attractive spin-orbit potential which becomes repulsive at still larger distances.



figure_5: phase-shift components for the L=2-waves (ref.20)

The values of the integrals for the D-waves lie in between those for the P-waves and the F-waves, as expected. Furthermore we notice that the behaviour of the iso-spin singlet potentials is quite different from the iso-spin triplet potentials. In figure 5 we see a dominantly attractive central - plus spin-spin potential, a weak repulsive tensor potential and a weak spin-orbit potential which is attractive at large distances and repulsive at shorter distances.

Although potentials of the Wigner-Eisenbud form had some success, it is quite well established that these can not fit the available data on elastic nucleon-

nucleon scattering satisfactorily (ref. 25). In particular, difficulties arise in trying to fit the spin-singlet and spin -triplet states 1S_0 and 1D_2 simultaneously. Okuba and Marshak (ref. 26) have considered the general form of the potential without the restriction to a linear \vec{k} -dependence. Without this limitation, two more terms are added to the form (57), namely :

$$(\vec{\sigma}^1 \cdot \vec{k})(\vec{\sigma}^2 \cdot \vec{k}) \quad (63a)$$

and the quadratic spin-orbit term :

$$(\vec{\sigma}^1 \cdot \vec{L})(\vec{\sigma}^2 \cdot \vec{L}) + (\vec{\sigma}^2 \cdot \vec{L})(\vec{\sigma}^1 \cdot \vec{L}) \quad (63b)$$

Futhermore the potentials v_i are not restricted to be functions of \vec{r}^2 only, but can also depend on \vec{k}^2 and \vec{L}^2 .

If one is only interested in elastic nucleon-nucleon scattering and not in off-shell effects, one of the terms may be dropped. The reason is that a linear relation for the terms of the potential can be found for on-shell processes. Usually one drops the term $(\vec{\sigma}^1 \cdot \vec{k})(\vec{\sigma}^2 \cdot \vec{k})$, as the matrix-elements (51) are easier to handle for \vec{L} -dependent than \vec{k} -dependent terms. Due to a theorem by Jost and Kohn (ref. 28), we can also drop the \vec{k}^2 -dependence of the potentials v_i for on-shell scattering, and assume \vec{r}^2 - and \vec{L}^2 -dependence only. This theorem states that the phase-shifts as a function of energy determine for each partial wave a \vec{k} -independent potential uniquely. At the moment the present empirical data do not require all the freedom implied in the Okubo-Marshak form (ref. 1).

7) Phenomenological Potentials

In this section we discuss briefly a few phenomenological potentials. The majority of the more recent potentials are manufactured by using meson field theoretical arguments. The historical development of the involvement of meson field theoretical arguments in the construction of the potentials is well-characterized by the Taketani program (ref. 29). According to this program we should start our understanding of the nucleon-nucleon interaction in terms of meson field theory for large separations of the two nucleons and progress towards shorter distances. The reason for this is that the situation becomes increasingly complicated for shorter distances. The range of the nucleonic interaction is about 2fm. It is generally felt that the interaction for distances larger than about 1.5fm is well understood (ref. 30). In this region the interaction is dominated by the exchange of one pion. We can construct potentials which are equivalent to the interaction caused by one pion exchange, in the sense that both lead to the same phase parameters. This potential is generally known as the One Pion Exchange Potential (OPEP). We discuss this and other meson field theoretical potentials in more detail in chapter III. We confine ourselves in this section to potentials in which meson-field theoretical arguments are restricted to the use of an OPEP-tail. These are potentials which coincide with OPEP for a nucleon-nucleon separation larger than about 1.5fm.

The superiority of phenomenological potentials which have an OPEP-tail and which are of the more general Okubo-Marshak type instead of a Wigner-Eisenbud form, is illustrated in table 4.

table 4

potential	reference	year	$\frac{1}{n} \chi^2$
Brückner-Gammel-Thaler	32	1958	106.
Yale	33	1962	3.81
Hamada-Johnston	34	1962	2.98
Reid (H.C.)	35	1968	2.72
Bressel et. al.	36	1969	2.13
Tabakin	37	1968	28.

A few phenomenological potentials and their goodness of fit per data point for a set of 648 p-p data is given in reference 20.

The potential constructed by Brückner, Gammel and Thaler (ref. 32), is a more successful one of the Wigner-Eisenbud type. It does not have the OPEP-asymptotic behaviour. The potentials of Yale (ref. 33) and Hamada-Johnston (ref. 34) are of the more general Okubo-Marshak form, and these do have an OPEP-tail. From table 4 one sees that the improvement of the fit is considerable. The Yale and Hamada-Johnston potentials differ in the way the extension of the Wigner-Eisenbud form is made. Both do not use the quadratic spin-orbit term 63b but instead, each in a different way, a more convenient comparable form. The functions v_i depend in the Yale form on both \vec{r}^2 and \vec{L}^2 and in the Hamada-Johnston potential only on \vec{r}^2 . Furthermore in the Yale potential one uses a larger number of parameters. These last two considerations together with the χ^2 -values made the Hamada-Johnston the more popular of the two.

Another method consists in defining a different potential for each separate $|JLS\rangle$ -state. This was tried by Reid (ref. 35). Table 4 shows that the improvement over the fit of the Hamada-Johnston potential is only small.

As we have seen in the last section, the empirical evidence shows that the potential is repulsive at short distances. This short range repulsion is handled in different ways in the various phenomenological potentials.

In the literature we may encounter infinitely repulsive cores and potentials in which a finite repulsion is used. The oldest model of the short range repulsion, introduced by Jastrow (ref. 38), uses an infinite square potential barrier at a certain distance r_c . This is usually referred to as a hard core. This hard core causes the wave function to vanish for distances smaller than r_c . This is convenient in calculations, because the repulsion can be accounted for by the use of the boundary condition $\Psi(r_c) = 0$. Another advantage of this description is that possible singularities in the potential at the origin are cut off by the core. A hard core of 0.48fm and 0.51fm respectively is used in both the Hamada-Johnston and the Yale potentials.

Bressel et. al. (ref. 36) have used a finite repulsion. They modified the Hamada-Johnston potential by replacing the hard core by a finite square barrier with a slightly larger radius (0.7fm) than the hard core. One expects to see a significantly different behaviour of the phase-shifts between a hard core and a 'soft core' model at energies of the order of the barrier height and higher. Bressel et. al. considered a barrier of 470 MeV. for the triplet even states and 650 MeV. or 670 MeV. for the other states. So for energies below 350 MeV. one expects

that the results of the models are comparable. The fit to the data that they obtained, as shown in table 4 is quite good. A difficulty in the calculations with their square barrier model is the discontinuity appearing at the edge. This discontinuity was avoided in the model for the repulsion of Rochleder and Erkelenz (ref. 39). They assumed a centrifugal type repulsion, up to a certain radius, within which a square potential was used. Of course if we use finite potentials, we should implicitly impose the condition that $u(r)$ vanishes for $r = 0$ (this does not apply if we are interested in the Born approximation only).

Smooth cores for an infinite repulsion have also been used. Wong (ref. 40) has employed an infinite repulsive Yukawa shape. He used this form for many-nucleon calculations in order to provide for extra binding energy. An advantage of finite and smooth potentials over hard core potentials is that, in the latter, the potential has large negative values just at the edge of the core. This is necessary in order to compensate for the repulsive core and to provide the correct low energy behaviour. These large negative values do not appear in finite and smooth potentials. We illustrate this in figure 6 in which a comparison is made between a hard core potential and a potential with a finite smooth repulsion.

Interesting with respect to hard cores is Baker's transformation (ref. 41, 42). Baker has shown how one can construct a potential which has a quadratic momentum dependence and no hard core from a potential which has a hard core, without necessarily being momentum dependent. He showed that the potentials are equivalent in the sense that they reproduce the same phase parameters.

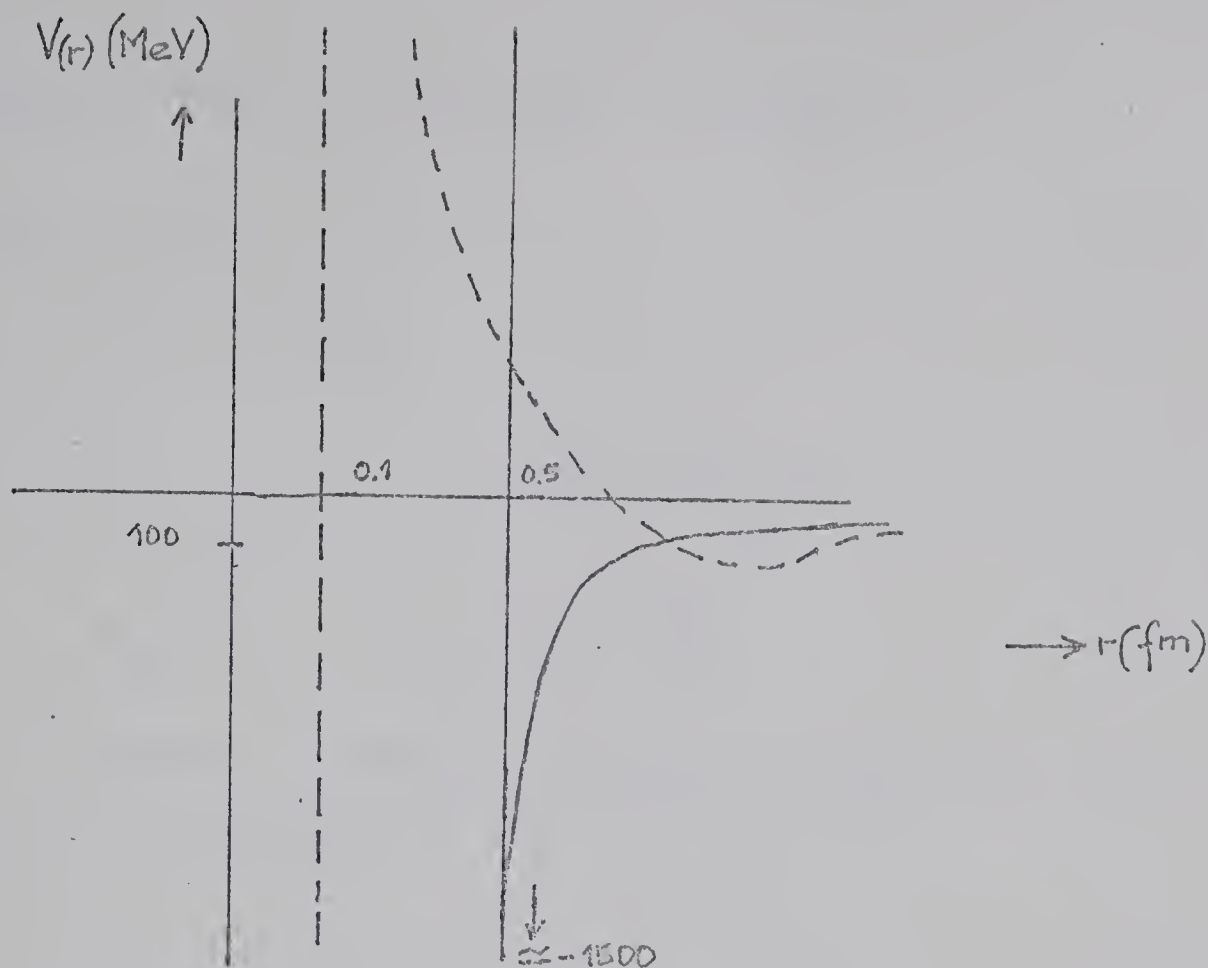


figure 6: hard core and soft core repulsion; the solid curve represents a hard core potential, the dotted curve a soft core potential (ref.20)

Green (ref. 43) has worked out a potential with a quadratic momentum dependence of the form :

$$V_1(r) + \frac{k^2}{2m} V_2(r) + V_2(r) \frac{k^2}{2m} \quad (64)$$

where the symmetrized form is required due to hermiticity. $k^2 = -\hbar^2 \Delta$ and $V_2(r)$ is short ranged and strongly repulsive. To illustrate how the corresponding Schrödinger problem is solved, we consider the spin independent case, for which no coupling between the partial waves occurs. The radial wave function is written as :

$$(1 + 2 V_2(r))^{-1/2} u_L(kr) \quad (65)$$

Upon inserting this quantity in the partial wave equation, we obtain the following equation for $u_L(kr)$:

$$\frac{d^2}{dr^2} u_L(kr) - L(L+1) r^{-2} u_L(kr) + k^2 u_L(kr) =$$

$$\frac{m}{\hbar^2} W_L(kr) u_L(kr) \quad (66a)$$

where :

$$W_L(kr) = V_1(r) (1 + 2 V_2(r))^{-1} - \left\{ \frac{dV_1(r)}{dr} (1 + 2 V_2(r))^{-1} \right\}^2 \frac{\hbar^2}{m} \\ + 2 V_2(r) (1 + 2 V_2(r))^{-1} \frac{\hbar^2 k^2}{m} \quad (66b)$$

In general, momentum dependent potentials are related to non-local potentials by (ref. 45) :

$$V(\vec{r}, \vec{k}) = \int d\vec{r}' U(\vec{r}, \vec{r}') e^{-i\vec{k} \cdot (\vec{r} - \vec{r}')} \quad (67)$$

Clearly the momentum dependence disappears if $U(\vec{r}, \vec{r}')$ is local :

$$U(\vec{r}, \vec{r}') = U(\vec{r}) \delta(\vec{r} - \vec{r}') \leftrightarrow V(\vec{r}, \vec{k}) = V(\vec{r}) \quad (68)$$

For the particular case that $U(\vec{r}, \vec{r}')$ is significantly different from zero only when $\vec{r} - \vec{r}'$ is small, and if $U(\vec{r}, \vec{r}')$ is a function of $|\vec{r} - \vec{r}'|$ only, we can show by expanding the exponential in (67) and retaining the zeroth and second terms only, that (67) coincides with (64). The momentum dependent form (64) is therefore a special case of a non-local potential.

Another special type of non-local potentials are separable potentials. These have the general form :

$$U(\vec{r}, \vec{r}') = u(\vec{r}) u(\vec{r}') \quad (69)$$

These have the great advantage over local potentials

that the Schrödinger equation for these can be solved analytically (ref. 45). The extension is sometimes made to a finite sum of separable potentials. A well-known potential of this kind is developed by Tabakin (ref. 37). He uses a sum of two separable potentials. As is shown in table 4, the fit to the data is not so good. As a separable potential is easy to handle, it is useful in many-body problems.

8) Concluding Remarks

On the basis of the existing information one can see that the present phenomenological potentials do not fit the scattering data as successfully as phase parameters do. A typical value of χ^2 per data point for phase-shift analysis is about 1, whereas for the more successful potentials this figure is somewhat above 2. However the number of parameters used in potentials is only about half of the number used in a phase parametrization (ref. 1). Nevertheless the number of free parameters in phenomenological potentials is still quite large (usually 30 to 50) (ref. 3), and since we are dealing with phenomenological potentials these parameters have hardly a physical meaning. Another important shortcoming is that the potential expression can not be determined uniquely. Although various forms lead to a reasonable fit, their off-shell behaviour can be quite different. Therefore using phenomenological potentials in many-nucleon problems is questionable.

A fundamental defect of potentials is that these

are not well-defined in a relativistic formalism. The rest mass of the nucleon is about 940 MeV. . If we are interested in laboratory kinetic energies up to 350 MeV. we should expect relativistic effects to become important. For 350 MeV. the ratio $(\frac{v}{c})^2$ is :

$$\left(\frac{v}{c}\right)^2 = p^2 c^2 / E^2 = \{ (Mc^2 + E_{\text{kin}})^2 - M^2 c^4 \} / (Mc^2 + E_{\text{kin}})^2 \approx \frac{1}{2} \quad (70)$$

Also at lower energies relativistic effects are important. The reason is that due to the strong short range repulsion, high momenta are generated in the wave packet.

The fact that a quadratic momentum dependence is inevitable for a satisfactory fitting of the data can be considered as a relativistic effect. This can be seen as follows : As was indicated in section 7, a momentum dependent potential can be described as a non-local potential. One may wonder how this non-locality comes about. A way to approach this is to transform to momentum space. One can easily verify that (ref. 46) :

$$U(\vec{r}, \vec{r}') = \frac{1}{(2\pi)^6} \int d\vec{q} d\vec{p} V(\vec{q}, \vec{p}) e^{i(\vec{q} \cdot \vec{R} + \vec{p} \cdot \vec{r})} \quad (71)$$

where : $\vec{p} = \vec{k} - \vec{k}'$: the momentum transfer

$\vec{q} = \frac{1}{2}(\vec{k} + \vec{k}')$: the average momentum of a nucleon before and after the collision in the c.m. frame

$$\vec{r} = \frac{\vec{r} + \vec{r}'}{2} \text{ and } \vec{R} = \vec{r} - \vec{r}'$$

From the form (71), it is seen that the non-locality of the potential is due to the dependence of the potential in momentum space on \vec{q} . If relativistic effects are significant then this is to be expected, because we can

no longer use the phase-space element $(2\pi)^{-6} d\vec{q} d\vec{p}$. This can be seen from the box normalization (ref. 47). Due to the Lorentz contraction, the box is contracted by a factor $\frac{M}{E}$, where M and E are mass and energy of the nucleon. Therefore we should use the phase-space elements :

$$\frac{M^2}{(2\pi)^6} \frac{d\vec{q}}{E_q} \frac{d\vec{p}}{E_p} \quad (72)$$

where $E_p = (M^2 + \vec{p}^2)^{1/2}$ $E_q = (M^2 + \vec{q}^2)^{1/2}$

This introduces a \vec{q} -dependence of the integrand in (68), other than from the exponential, so we should use a non-local potential.

By using relativistic phase-space elements, we can partially take relativistic effects into account (ref. 48). A rigorous relativistic formalism is offered by field theory and dispersion relations, which will be discussed in the next chapter. These alternatives to the potential model approach are not only of interest from a more fundamental point of view, but also from practical considerations due to the problems connected with the phenomenological potentials.

CHAPTER II

MESON FIELD THEORETICAL METHODS

1) Introduction

In chapter I we discussed the use of phenomenological potentials for the nucleon-nucleon interaction. These potentials do not offer an explanation of the interaction, but rather describe the empirical data. A more fundamental approach of the interaction is based on the assumption that it is due to the exchange of particles. This hypothesis was proposed by Yukawa (ref. 49) in 1934.

Yukawa was led by analogies between the nucleon-nucleon interaction and the electro-magnetic interaction. In the latter, the interaction between charged particles is due to the exchange of photons. Contrary to the electro-magnetic force however, the nuclear force is short ranged. Yukawa explained this feature by assuming that the mesons, which carry the nucleon-nucleon interaction, are massive, whereas the photons in the electro-magnetic case have a zero mass. That this assumption leads to a short range interaction can be illustrated by using the uncertainty principle (ref. 4).

The energy E of a meson emitted by a nucleon is:

$$E = \sqrt{p^2 c^2 + m^2 c^4} \quad (1)$$

where m and p are mass and momentum of the meson respectively; c : velocity of light

The meson emission by the nucleon is allowed by the principle of conservation of energy, as long as the life

time of the meson τ does not exceed : $\tau_{\max} = \frac{\hbar}{E}$.

Futhermore we suppose that $p^2 c^2 < m^2 c^4$. We see that the meson can not travel a distance larger than x_{\max} :

$$x_{\max} = c\tau_{\max} = \frac{\hbar}{mc} \quad (2)$$

From (2) we observe that the range of the interaction mediated by a particle of mass m is inversely proportional to the mass m .

A meson, the pion which interacts strongly with nucleons and which possesses a mass as predicted by Yukawa, was discovered in 1947 (ref. 50). This discovery added much to the credibility of the Yukawa hypothesis.

The procedures that one used in meson field theories of the nucleon-nucleon interaction were quite similar to those of the very successful quantum field theory of electrodynamics. However the modeling of the meson field theory after quantum electrodynamics is not without problems. In quantum electrodynamics, extensive use is made of perturbative treatments of some postulated form of the interaction. If one proceeds to higher orders, one encounters divergence problems. It appeared possible to deal with the divergences in quantum electrodynamics by the use of renormalization techniques. In the case of the nuclear force, the interaction is much stronger than in electrodynamics. The application of these perturbative methods to the nucleon-nucleon interaction is therefore very questionable. Futhermore one can not use the renormalization technique for some forms of the interaction.

Nevertheless it turned out that by applying these methods to the nucleon-nucleon interaction, one could

obtain a reasonable agreement with the empirical data (ref. 3). From a practical point of view one may therefore use these methods.

On the other hand, one may try to develop techniques which do not suffer from these fundamental defects and which are more suitable to the specific case of the nuclear force. Efforts in this direction have led to the development of the axiomatic formulations of field theory (ref. 51) , in which the perturbative methods were abandoned. These axiomatic field theories in turn inspired the development of the dispersion relation technique. This technique has proven to be useful for strong interactions (ref. 52).

In this chapter we discuss a few theoretical methods that are used to describe the nuclear force, starting from the assumption that this force is due to the exchange of mesons. In section 2 we outline the Lagrangian formulation of field theory. Covariant perturbation theory and Feynman graphs are briefly discussed in section 3. The Bethe-Salpeter equation is considered in section 4. Section 5 is devoted to a short discussion on the applicability of Lagrangian field theory to the nuclear force. The S-matrix is discussed in section 6. In the next section we consider dispersion relations in some detail. One is not able to give a rigorous proof of these relations from field theory. In order to use these we have to make additional assumptions. These additional assumptions are conveniently formulated in the Mandelstam representation, which is the topic of the 8th section. A few concluding remarks concerning dispersion relations are given in section 9. Furthermore an appendix on the helicity-state representation is added to this chapter.

2) The Lagrangian Formulation

From a meson field theoretical viewpoint, the nuclear force arises as a result of an exchange of mesons which are created at one nucleon and annihilated at the other. In such description we can therefore avoid the notion of an 'action at a distance', which is not allowed in a relativistic theory. The formalism should describe the phenomena of particle creation and - annihilation. The theory can be considered as the quantum mechanics of a system with an infinite number of degrees of freedom. We associate to each kind of particle a field, which is a function in space-time and which can have more than one component. The degrees of freedom are formed by the values of the field at each space-time point.

In the Lagrangian formulation of field theory (ref. 53) , introduced by Pauli, Heisenberg and others in the late twenties, one postulates a form for the Lagrangian density, which is a function of the fields. From this Lagrangian density one can derive the field equations by means of a variational principle. In the case of non-interacting Bosons this is the Klein-Gordon equation. For non-interacting Fermions the associated field should satisfy the Dirac equations. Knowing the Lagrangian density we can define the conjugate momenta for the fields. The theory can subsequently be quantized by interpreting the fields and their conjugate momenta as operators, which satisfy the quantum mechanical commutation relations. In order to obtain the correct statistics, we must use commutation relations for the Boson fields and anti-commutation relations for the Fermion fields.

Contact with the particle interpretation of the fields can be established by considering the Fourier

decomposition of the field operators. We shall first discuss Boson fields and then Fermion fields. It will be convenient, although by no means essential, to normalize the fields in a large, but finite, volume V . In that way it is easier to handle the denumerability of the number of degrees of freedom.

The Fourier expansion of a Boson field $\phi(x)$ can be written as :

$$\phi(x) = V^{-1/2} \sum_{p, p_0=E_p} (2p_0)^{-1/2} \{a_p e^{ipx} + a_p^\dagger e^{-ipx}\} \quad (3)$$

In this expression is p the 4-momentum and x a space-time point. $E_p = (m^2 + \vec{p}^2)^{1/2}$ is the energy and m is the mass of the Boson. Furthermore $px \equiv \vec{p} \cdot \vec{x} - p_0 x_0$. The appearance of the factor $(V2p_0)^{-1/2}$ is due to the relativistically invariant normalization that we assumed. If we consider the limit V goes to infinity, we can make the well-known replacement :

$$V^{-1/2} \sum_p \frac{1}{2p_0} \rightarrow (2\pi)^{-3} \int \frac{d^3p}{2p_0} \quad (4)$$

From the commutation relations between the field operator and its conjugate momentum follows that the operators a_p and a_p^\dagger obey the commutation relations :

$$[a_{p_i}, a_{p_j}] = [a_{p_i}^\dagger, a_{p_j}^\dagger] = 0 \quad (5a)$$

$$[a_{p_i}, a_{p_j}^\dagger] = \delta_{p_i p_j} \quad (5b)$$

The relations (5a) and (5b) allow one to interpret a_p^\dagger as an creation operator and a_p as an annihilation operator, for a Boson with 4-momentum p :

$$a_{p_i} |n_1 \dots n_i \dots\rangle = \sqrt{n_i} |n_1 \dots n_i - 1 \dots\rangle \quad (6a)$$

$$a_{p_j}^\dagger |n_1 \dots n_i \dots\rangle = \sqrt{n_i + 1} |n_1 \dots n_i + 1 \dots\rangle \quad (6b)$$

In the Fock space vector $|n_1 \dots n_i \dots\rangle$, n_i denotes the number of Bosons with the 4-momentum p_i . It is often useful to decompose the Boson field operator $\phi(x)$ into a creation part $\phi^-(x)$ and an annihilation part $\phi^+(x)$:

$$\begin{aligned} \phi(x) &= \phi^+(x) + \phi^-(x) \quad ; \quad \phi^+(x) = V^{-1/2} \sum_p (2p_0)^{-1/2} a_p e^{ipx} \\ &\quad ; \quad \phi^-(x) = V^{-1/2} \sum_p (2p_0)^{-1/2} a_p^\dagger e^{-ipx} \end{aligned} \quad (7)$$

The time-ordered product of the field operators $\phi(x)$ and $\phi(x')$ is defined by :

$$P\{\phi(x), \phi(x')\} = \begin{cases} \phi(x) \phi(x') & \text{if } x_0 > x'_0 \\ \phi(x') \phi(x) & \text{if } x'_0 > x_0 \end{cases} \quad (8)$$

It can be shown that :

$$\begin{aligned} \langle 0 | P\{\phi(x), \phi(x')\} | 0 \rangle &= \frac{-i}{(2\pi)^4} \int dp e^{ip(x-x')} (p^2 + m^2 - i\eta)^{-1} \equiv \\ &\equiv i\Delta_F(x-x') \end{aligned} \quad (9)$$

In (9) $|0\rangle$ denotes the vacuum. The function $\Delta_F(x-y)$ is called the propagator for the Boson. The reason for this can be seen from :

$$(\Box^2 - m^2) \Delta_F(x-y) = \delta(x-y) \quad ; \quad \Box^2 \equiv \Delta - \frac{\partial^2}{\partial t^2} \quad (10)$$

(10) is easily proven by substitution of (9) into the Klein-Gordon equation. Having introduced the necessary ingredients for Boson fields, we next turn to Fermion fields. The description of Fermion fields is somewhat more involved, as we are dealing with spinor fields. For spin- $\frac{1}{2}$ particles, the spinor field has four components.

The Fourier expansion of the spinor field $\Psi(x)$ is usually written as (ref. 56) :

$$\begin{aligned} \Psi(x) = & \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \sum_s \left(\frac{M}{E_k}\right)^{1/2} \{a_{s\mathbf{k}} u_s(\mathbf{k}) e^{i\mathbf{k}x} + \\ & + b_{s\mathbf{k}}^\dagger v_s(\mathbf{k}) e^{-i\mathbf{k}x}\} \end{aligned} \quad (11)$$

In (11) \mathbf{k} denotes the 4-momentum; $E_k = (M^2 + k^2)^{1/2}$ the energy, M the mass, and s the spin state of the Fermion.

The expansion of the Dirac adjoint $\bar{\Psi}(x) = \Psi^\dagger(x) \gamma_0$ is :

$$\begin{aligned} \bar{\Psi}(x) = & \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \sum_s \left(\frac{M}{E_k}\right)^{1/2} \{a_{s\mathbf{k}}^\dagger \bar{u}_s(\mathbf{k}) e^{-i\mathbf{k}x} + \\ & + b_{s\mathbf{k}} \bar{v}_s(\mathbf{k}) e^{i\mathbf{k}x}\} \end{aligned} \quad (12)$$

In the solution of the Dirac equation $u_s(\mathbf{k})$ denotes a positive energy spinor and $v_s(\mathbf{k})$ a negative energy spinor. $a_s(\mathbf{k})$ and $a_s^\dagger(\mathbf{k})$ can be interpreted as annihilation and creation operators for the spin- $\frac{1}{2}$ particles and $b_s(\mathbf{k}), b_s^\dagger(\mathbf{k})$ as the annihilation and creation operators for their antiparticles.

We decompose $\Psi(x)$ and $\bar{\Psi}(x)$ into positive and negative frequency parts :

$$\Psi(x) = \Psi^+(x) + \Psi^-(x) \quad (13)$$

$$\bar{\Psi}(x) = \bar{\Psi}^+(x) + \bar{\Psi}^-(x) \quad (14)$$

In this decomposition :

$$\begin{aligned} \bar{\Psi}^-(x) &: \text{creates particles} \\ \Psi^+(x) &: \text{annihilates particles} \\ \Psi^-(x) &: \text{creates antiparticles} \\ \bar{\Psi}^+(x) &: \text{annihilates antiparticles} \end{aligned} \quad (15)$$

The propagator $S_F(x-x')$ for spin- $\frac{1}{2}$ particles should satisfy :

$$(i\gamma^\mu \partial_\mu + M) S_F(x-x') = \delta(x-x') \quad (16)$$

From .

$$(i\gamma^\mu \partial_\mu - M)(i\gamma^\mu \partial_\mu + M) = \square^2 - M^2 \quad (17)$$

we find a relation between $S_F(x-x')$ and $\Delta_F(x-x')$:

$$\begin{aligned} S_F(x-x') &= (i\gamma^\mu \partial_\mu - M) \Delta_F(x-x') = \\ &= \frac{1}{(2\pi)^4} \int dk \frac{\gamma^\mu k_\mu + M}{k^2 + M^2 - i\eta} e^{ik(x-x')} \end{aligned} \quad (18)$$

Often this is symbolically written as :

$$S_F(x-x') = \frac{1}{(2\pi)^4} \int dk (M - \gamma^\mu k_\mu - i\eta)^{-1} e^{ik(x-x')} \quad (19)$$

Similar expressions for $S_F(x-x')$ of (11) in the case of Bosons are :

$$\langle 0 | T\{\Psi(x), \bar{\Psi}(x')\} | 0 \rangle = iS_F(x-x') \quad (20)$$

$$\langle 0 | T\{\bar{\Psi}(x), \Psi(x')\} | 0 \rangle = -iS_F(x'-x) \quad (21)$$

In these expressions $T\{.,.\}$ is the Wick chronological product, which differs from the time-ordered product $P\{.,.\}$ for Fermion fields in that :

$$\begin{aligned} T\{\Psi(x), \Psi(x')\} &= \varepsilon(x_0 - x'_0) P\{\Psi(x), \Psi(x')\}; \\ \varepsilon(x_0 - x'_0) &= \begin{cases} +1 & \text{if } x_0 > x'_0 \\ -1 & \text{if } x'_0 > x_0 \end{cases} \end{aligned} \quad (22)$$

These remarks on spin- $\frac{1}{2}$ fields are sufficient to serve as a convenient reference in the following. We finish this section by fixing a convention concerning the Dirac operators γ^μ . The following, frequently used representation will be adopted (ref. 3) :

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} ; \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} ; \quad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

where :

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (23)$$

Furthermore we shall use the quantity $\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$. For quantities like $\bar{\Psi}_\alpha (\gamma^\mu)_{\alpha\beta} (\gamma^5)_{\beta\gamma} \Psi_\gamma$, we shall use the abbreviated notation $\bar{\Psi} \gamma^\mu \gamma^5 \Psi$.

The Lagrangian density can be written as the sum of the free field contributions and an interaction part. The required strong-interaction invariances imply that the Lagrangian density is a scalar with respect to

Lorentz transformations and rotations in iso-space. Furthermore as required for an energy density, the interaction Lagrangian density should be hermitian. In the simplest forms of the interaction Lagrangian density the meson fields or their derivatives appear linearly. The first case is called direct coupling and the second derivative coupling. The form of the interaction Lagrangian density for a number of iso-scalar mesons is given in table 5.

table 5 : interaction Lagrangian densities

type of meson (J^π)	direct coupling	derivative coupling
scalar (0^+)	$-\sqrt{(4\pi)} g_s \bar{\Psi} \Psi \phi$	$-\sqrt{(4\pi)} \frac{f_s}{m} \bar{\Psi} \gamma^\mu \Psi \partial_\mu \phi$
pseudo scalar (0^-)	$-\sqrt{(4\pi)} i g_{ps} \bar{\Psi} \gamma^5 \Psi \phi$	$-\sqrt{(4\pi)} \frac{f_{ps}}{m} \bar{\Psi} \gamma^5 \gamma^\mu \Psi \partial_\mu \phi$
vector (1^-)	$-\sqrt{(4\pi)} g_v \bar{\Psi} \gamma^\mu \Psi \phi_\mu$	$-\sqrt{(4\pi)} \frac{f_v}{m} \bar{\Psi} \sigma^{\mu\nu} \Psi (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu)$

The factors $-\sqrt{(4\pi)}$ are conventionally pulled out. This will be convenient in the calculations. The factors g_i and f_i are the coupling constants. They determine the strength of the coupling. In the derivative coupling, the inverse of the meson mass $\frac{1}{m}$ is written in front, in order to make f_i dimensionless. In the various forms for the interaction Lagrangian \mathcal{L}_I in table 5, the coordinate dependences of the fields are suppressed. The forms for iso-vector mesons are found by simply replacing the meson fields ϕ by $\vec{\tau} \cdot \vec{\phi}$, where $\vec{\phi}$

is the iso-vector meson field and $\vec{\tau}$ denotes the nucleon iso-spin.

In the formalism we rather use the Hamilton density instead of the Lagrangian density. The relation between the two is :

$$\mathcal{H}_I(x) = \frac{\partial \mathcal{L}_I(x)}{\partial \dot{\phi}} \cdot \dot{\phi} - \mathcal{L}_I(x) \quad (24)$$

For direct couplings we find :

$$\mathcal{H}_I(x) = - \mathcal{L}_I(x) \quad (25)$$

For derivative couplings we notice from (24) that $\mathcal{H}_I(x)$ not only differs in sign from $\mathcal{L}_I(x)$, but also in that the summation over the variables x_μ runs over the position variables x_1 , x_2 and x_3 only.

3) Covariant Perturbation Theory

In this section we show how the scattering amplitude can be calculated from a given form of $\mathcal{H}_I(x)$. The formalism, developed essentially by Dyson (ref. 54) and Feynman (ref. 55), that will be considered, allows us to write the scattering amplitude as a perturbation expansion in powers of the coupling constant.

In our notation, $\phi(t)$ represents the state of the system of interacting nucleons and mesons as a function of time. In the interaction picture, the equation of motion can be written as :

$$i \frac{\partial}{\partial t} \phi(t) = H_I(t) \phi(t) \quad (26)$$

In this equation, $H_I(t)$ is the interaction Hamiltonian, which is related to the Hamilton density $\mathcal{H}_I(x)$ by :

$$H_I(t) = \int d^3x \mathcal{H}_I(x) \quad (27)$$

The scattering operator S is defined by :

$$\phi(+\infty) = S \phi(-\infty) \quad (28)$$

$\phi(+\infty)$ and $\phi(-\infty)$ represent the state of the system respectively long after and long before the interaction takes place.

By integration and iteration of (26) and the use of the definitions (27) and (28), we can write the S-operator as an expansion :

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} dx_1 \dots dx_n P\{\mathcal{H}_I(x_1) \dots \mathcal{H}_I(x_n)\} \quad (29)$$

In this expansion, $P\{.,.\}$ is the time-ordered product, which is an obvious extension of the definition in (8). The calculation of matrix elements of operators consisting of creation and annihilation operators can be simplified if we write the operators in a normal ordering. In the normal ordering, the annihilation operators stand to the right of the creation operators in a product.

$\mathcal{H}_I(x)$ can be written in an abstract way as $\bar{\Psi}\Gamma\Psi\phi$, in which possible indices are suppressed. If we decompose the fields $\bar{\Psi}$, Ψ and ϕ in an annihilation and creation part, we can write $\mathcal{H}_I(x)$ as the sum of normal ordered products, by using the appropriate (anti)commutation relations. We denote the normal ordering of $\bar{\Psi}\Gamma\Psi\phi$ by $N\{\bar{\Psi}\Gamma\Psi\phi\}$. Substitution in the S-operator expansion gives :

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} dx_1 \dots dx_n T\{N\{\bar{\Psi}\Gamma\Psi\phi\}_{x_1} \dots N\{\bar{\Psi}\Gamma\Psi\phi\}_{x_n}\} \quad (30)$$

In this formula, the time-ordered product $P\{.,.\}$ is replaced by the Wick chronological product $T\{.,.\}$. This is allowed because the Fermion fields enter the equation in pairs, each of which refers to the same time. The notation $\{\bar{\Psi}\Gamma\Psi\phi\}_{x_1}$ is short-hand for $\bar{\Psi}(x_1)\Gamma\Psi(x_1)\phi(x_1)$.

In (30) we obviously do not have an overall normal ordering, but rather a time ordering. This time-ordered product can be written in a normal order according to the theorem of Wick. To introduce the Wick theorem we consider the following :

The simplifying feature of a normal ordering is that the vacuum expectation value for a normal ordered product vanishes. This leads to :

$$T\{A(x_1), B(x_2)\} = N\{A(x_1), B(x_2)\} + \langle 0 | T\{A(x_1), B(x_2)\} | 0 \rangle \quad (31)$$

In this expression $A(x_1)$ and $B(x_2)$ are two arbitrary fields. The vacuum expectation value $\langle 0 | T\{A(x_1), B(x_2)\} | 0 \rangle$ is called the contraction of $A(x_1)$ and $B(x_2)$, and often one uses the notation :

$$\underbrace{A(x_1) B(x_2)}_{12} \equiv \langle 0 | T\{A(x_1), B(x_2)\} | 0 \rangle \quad (32)$$

More generally we can define a similar notation for the product of more than two field operators by :

$$\underbrace{N\{ABCDEF\dots\}}_{12\dots} \equiv (-1)^P \underbrace{AF}_{12} \underbrace{BD}_{34} N\{CE\dots\} \quad (33)$$

In (33) we dropped the explicit coordinate dependences. P denotes the number of interchanges of the Fermion fields required to change to the order $AFBDCE\dots$.

The Wick theorem is a generalization of (31). It states in terms of the notation (33) :

$$\begin{aligned} T\{ABCD\dots YZ\} = & N\{ABCD\dots YZ\} + \dots : \text{no contractions} \\ & + \underbrace{N\{ABCD\dots YZ\}}_{12} + \underbrace{N\{ABCD\dots YZ\}}_{34} + \dots : \text{one-contractions} \\ & + \underbrace{N\{ABCD\dots YZ\}}_{1234} + \underbrace{N\{ABCD\dots YZ\}}_{1324} + \dots : \text{two-contractions} \\ & + \dots \text{ etc} \end{aligned} \quad (34)$$

Using the Wick theorem we can write the integrands in the S -operator expansion as a sum of normal products :

$$S = \sum_n S_n ; S_n = \frac{(-i)^n}{n!} \int \dots \int dx_1 \dots dx_n T\{N(\bar{\Psi}\Gamma\Psi\phi)_{x_1} \dots N(\bar{\Psi}\Gamma\Psi\phi)_{x_n}\} \quad (35)$$

The integrals in S_n run from minus-infinity to plus-infinity. S_0 , S_1 and S_2 are respectively :

$$S_0 = I : \text{the identity operator} \quad (36)$$

$$S_1 = -i \int dx N(\bar{\Psi} \Gamma \Psi \phi)_x \quad (37)$$

$$\begin{aligned}
 S_2 = & -\frac{1}{2} \int dx_1 \int dx_2 [N\{(\bar{\Psi} \Gamma \Psi \phi)_{x_1} (\bar{\Psi} \Gamma \Psi \phi)_{x_2}\} + N\{(\bar{\Psi} \Gamma \Psi \phi)_{x_1} (\bar{\Psi} \Gamma \Psi \phi)_{x_2}\} + \\
 & + N\{(\bar{\Psi} \Gamma \Psi \phi)_{x_1} (\bar{\Psi} \Gamma \Psi \phi)_{x_2}\} + N\{(\bar{\Psi} \Gamma \Psi \phi)_{x_1} (\bar{\Psi} \Gamma \Psi \phi)_{x_2}\} + \\
 & + N\{(\bar{\Psi} \Gamma \Psi \phi)_{x_1} (\bar{\Psi} \Gamma \Psi \phi)_{x_2}\} + N\{(\bar{\Psi} \Gamma \Psi \phi)_{x_1} (\bar{\Psi} \Gamma \Psi \phi)_{x_2}\} + \\
 & + N\{(\bar{\Psi} \Gamma \Psi \phi)_{x_1} (\bar{\Psi} \Gamma \Psi \phi)_{x_2}\} + N\{(\bar{\Psi} \Gamma \Psi \phi)_{x_1} (\bar{\Psi} \Gamma \Psi \phi)_{x_2}\}] \quad (38)
 \end{aligned}$$

One can easily verify that $\bar{\Psi}\bar{\Psi}=\Psi\Psi=\phi\phi=\Psi\phi=\bar{\Psi}\phi=0$. Therefore those contractions are left out of (38).

In the process of nucleon-nucleon scattering, only contributions to the scattering matrix element of the form :

$$\langle 2 \text{ nucleons; } 0 \text{ mesons} | S | 2 \text{ nucleons; } 0 \text{ mesons} \rangle \quad (39)$$

can arise from S-operator terms S_n , for n is even. Furthermore in the decomposition of S_n , only terms which contain powers of the combination $\bar{\Psi}\Psi\bar{\Psi}\Psi$ can give a non-vanishing contribution to the interaction.

Therefore the lowest order contribution to the scattering amplitude comes from S_2 . The only term in S_2 which contributes to the interaction, is :

$$\begin{aligned}
& -\frac{1}{2} \iint dx_1 dx_2 N \{ (\bar{\Psi} \Gamma \Psi \phi)_{\substack{x_1 \\ \boxed{}}} (\bar{\Psi} \Gamma \Psi \phi)_{x_2} \} \\
& = -\frac{i}{2} \iint dx_1 dx_2 N \{ (\bar{\Psi} \Gamma \Psi \phi)_{x_1} (\bar{\Psi} \Gamma \Psi \phi)_{x_2} \} \Delta_F(x_1 - x_2)
\end{aligned}
\tag{40}$$

In (40) we used (9).

The only term in the normal product which contributes to the matrix element (40) is :

$$S_2^{NN} = \frac{i}{2} \iint dx_1 dx_2 \bar{\Psi}^-(x_1) \bar{\Psi}^-(x_2) \Gamma \Psi^+(x_1) \Gamma \Psi^+(x_2) \Delta_F(x_1 - x_2)
\tag{41}$$

The integrations over x_1 and x_2 can be performed after the substitution of the Fourier expansions for the field operators (11) and (12) and the representation (9) of $\Delta_F(x-x')$. After a straight forward calculation we arrive at :

$$\begin{aligned}
S_2^{NN} &= \frac{i(2\pi)^4}{2V^2} M^2 \sum_{\substack{k_\alpha k'_\alpha \\ k_\beta k'_\beta}} (E_{k_\alpha} E_{k'_\alpha} E_{k_\beta} E_{k'_\beta})^{-1/2} \delta(k'_\alpha + k'_\beta - k_\alpha - k_\beta) \cdot \\
&\cdot [\bar{u}_{k'_\alpha} \Gamma u_{k_\alpha}] [\bar{u}_{k'_\beta} \Gamma u_{k_\beta}] a_{k'_\alpha}^\dagger a_{k'_\beta}^\dagger a_{k_\alpha} a_{k_\beta} (m^2 + (k'_\alpha - k_\alpha)^2 - i\eta)^{-1}
\end{aligned}
\tag{42}$$

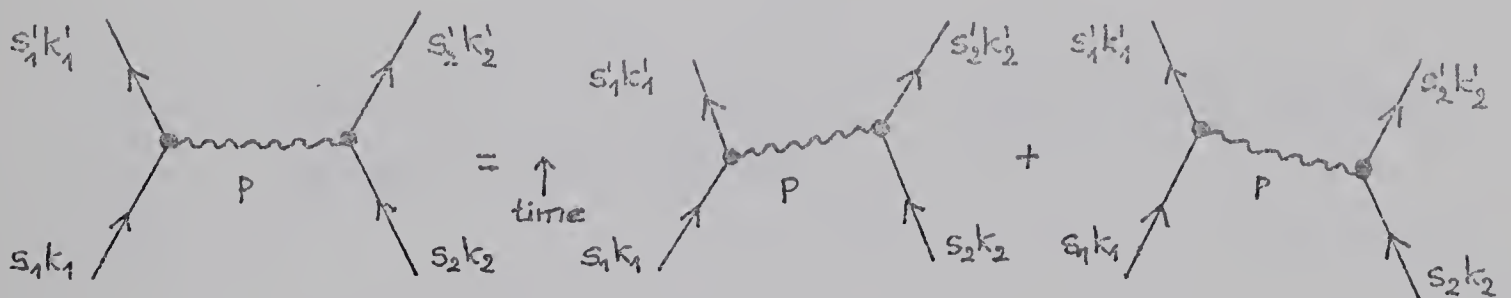
In (42) we suppressed the spin indices. The δ -function secures the conservation of energy momentum. It enters the calculation through the integration over p of the factor $\delta(p - k'_\alpha + k_\alpha) \delta(p - k'_\beta + k_\beta)$. The appearance of these factors is due to the integration over x_1 and x_2 , which occur in the exponentials. We consider the matrix element:

$$C_2(k'_1 s'_1 k'_2 s'_2; k_1 s_1 k_2 s_2) \equiv \langle k'_1 s'_1 k'_2 s'_2 | S_2^{NN} | k_1 s_1 k_2 s_2 \rangle
\tag{43}$$

Two terms in the expansion of S_2^{NN} contribute, and these contributions are equal. The explicit form of the matrix element is :

$$C_2 = i(2\pi)^4 \delta(k_1' + k_2' - k_1 - k_2) M^2 (E_{k_1} E_{k_1'} E_{k_2} E_{k_2'})^{-1/2} (2\pi)^{-6} \cdot [\bar{u}_{s_1}(k_1') \Gamma u_{s_1}(k_1)] [\bar{u}_{s_2}(k_2') \Gamma u_{s_2}(k_2)] (m^2 + (k_1' - k_1)^2 - i\eta)^{-1} \quad (44)$$

In (44) the limit $V \rightarrow \infty$ was taken, and the variables in C_2 are suppressed. C is known as the Feynman amplitude and C_2 is the second order contribution to it. It should be mentioned that in the literature, modified forms are frequently used. The Feynman graph that corresponds to (44) is shown in figure 7. The two terms in (42), that contribute, can be interpreted as two different time orderings.








figure_7:the Feynman graph for the Born term

These graphs are very convenient for a quick evaluation of complex coupling schemes, and for the understanding of the underlying physics of the mathematical formalism. By following a set of rules (ref. 56), we can immediately write down the contribution to the Feynman amplitude

of a given graphical representation. The rules, which are of importance for the nucleon-nucleon scattering problem, are listed in table 6.

table 6 : rules for Feynman graphs

graphical element	name of the element	physical process	expression
	external nucleon line	emitted nucleon	$(2\pi)^{-3/2} \left(\frac{M}{E_k}\right)^{1/2} \bar{u}_S(k)$
	external nucleon line	absorbed nucleon	$(2\pi)^{-3/2} \left(\frac{M}{E_k}\right)^{1/2} u_S(k)$
	internal meson line	virtual meson	$\frac{i}{(2\pi)^4} \int d^4p (m^2 + p^2 - i\eta)^{-1}$
	internal nucleon line	virtual nucleon	$\frac{i}{(2\pi)^4} \int d^4k (M - \gamma^\mu k_\mu - i\eta)^{-1}$
	vertex	interaction	$-\Gamma (2\pi)^4 \delta(k' - k - p)$

We apply these rules to find the contribution to the Feynman amplitude, represented by the box diagram (figure 8). After carrying out the four integrations we get :

$$\begin{aligned}
C_4 &= (2\pi)^{-6} \delta(k_1 + k_2 - k'_1 - k'_2) M^2 (E_{k_1} E_{k'_1} E_{k_2} E_{k'_2})^{-1/2} \\
&\cdot [\bar{u}_{s'_1}(k'_1) \Gamma u_{s_1}(k_1)] [\bar{u}_{s'_2}(k'_2) \Gamma u_{s_2}(k_2)] \\
&\cdot \int dp \{ (m^2 + (\frac{1}{2}k_1 - \frac{1}{2}k_2 - p)^2 - i\eta) (m^2 + (\frac{1}{2}k'_1 - \frac{1}{2}k'_2 - p)^2 - i\eta) \\
&\cdot (M - \gamma^\mu (\frac{1}{2}K - p)_\mu - i\eta) (M - \gamma^\mu (\frac{1}{2}K + p)_\mu - i\eta) \}^{-1}
\end{aligned} \tag{45}$$

where : $K = k_1 + k_2$ and $p = \frac{1}{2}(p_1 - p_2)$, see figure 8.

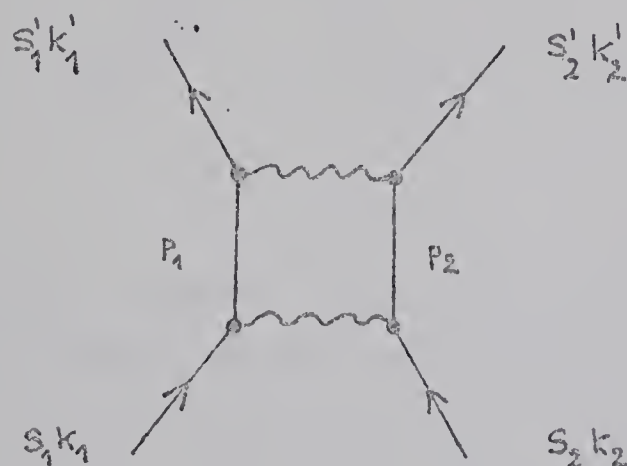


figure 8:
the box diagram

We notice that in the term S_4^{NN} there appears a factor $\frac{1}{4!}$, while in the formula (45) this seems not to be there. The reason for this is that the box diagram is in fact a summation of $4!$ time-ordered diagrams (figure 9):

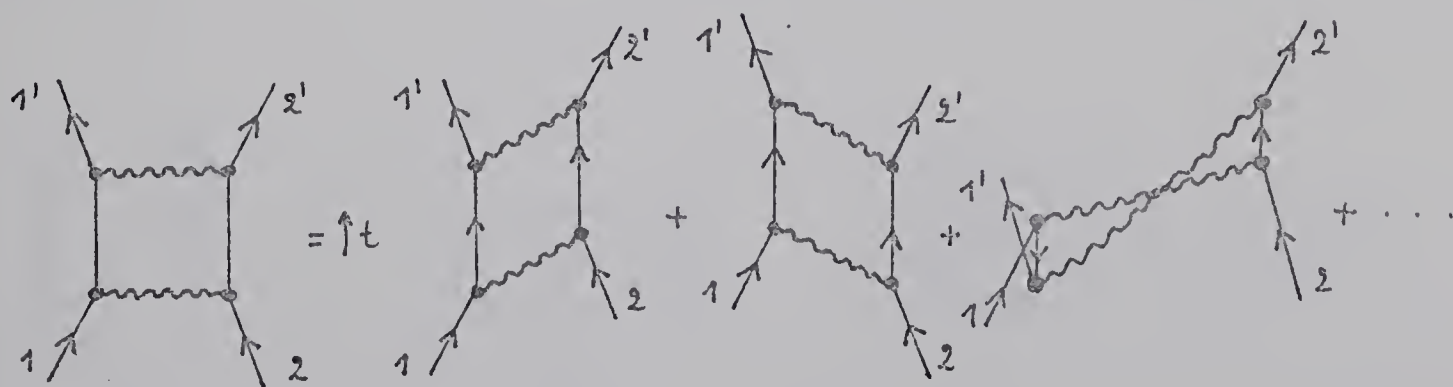


figure 9: time orderings in the box diagram

One sees in figure 9 that in the intermediate state we may encounter antinucleons. Of course the box diagram is not the only contribution to the fourth order Feynman amplitude. Another possibility is formed by the crossed meson exchange diagram (figure 10), in which also four nucleon meson interactions occur.

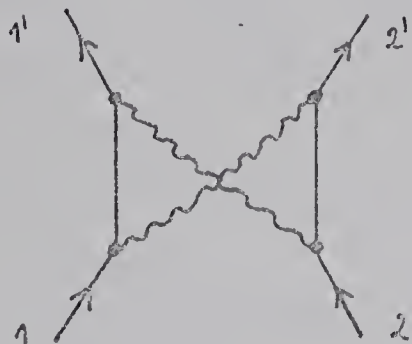


figure 10:
the crossed diagram

By first drawing all possible topologically non-equivalent Feynman graphs and subsequently writing down the associated expressions for the Feynman amplitude, we can in principle calculate the Feynman amplitude for each order.

Problems may arise from possible divergent integrals. Two other fourth order diagrams which give rise to divergences (ref. 4). We draw these in the figures 11a and 11b.



figure 11:
fourth order diagrams
that lead to diver-
gences

In figure 11a, we consider the emission and subsequent absorption of a virtual meson at one of the two nucleons. We notice that this process also occurs in the case of a free nucleon. This virtual meson production has the effect of increasing the nucleon energy, or equivalently its mass. This situation raises questions about the correctness of the decomposition of the Hamiltonian into a free and an interaction part. In the Hamiltonian of the so called free nucleon, the virtual meson contributions are implicitly taken into account by the value of the nucleon mass. We therefore should write the nucleon mass as $M = M_0 + \delta M$, in which the part δM is due to the interaction with the virtual mesons. If, in the free Hamiltonian, the mass M appears in the form $\bar{M}\Psi\Psi$, we should therefore subtract the quantity $\delta M\bar{\Psi}\Psi$ from the interaction Hamiltonian. It has been shown that for a correctly chosen value of δM , the divergence problem can be solved (ref. 56).

This technique is called the renormalization procedure. Similar techniques can be used to deal with the divergences that appear in the interactions represented by figure 11b in which a virtual nucleon-antinucleon pair appears. The procedure prescribes the use of so-called renormalized values for the masses of nucleons and mesons to incorporate the primary effects of the diagrams 11a and 11b (ref. 4).

The renormalization procedure can not be used for all forms of the interaction Hamiltonian. For instance it does not apply to derivative coupling with pseudo scalar fields. Renormalizable theories are those in which the renormalization technique can be carried out to all orders. It has been shown that theories of mesons interacting with nucleons are renormalizable for non-derivative couplings (ref. 57).

4) The Bethe-Salpeter Equation

In section 3 we considered the perturbative expansion of the scattering operator S , or equivalently of the transition operator $T=S-I$, in orders of the interaction Hamiltonian. This expansion is quite similar to the Born expansion in ordinary quantum mechanics. By iteration of the Lippmann-Schwinger equation, which may symbolically be written as :

$$T = V + VGT \quad (46)$$

where: V : potential ; G : Green function

one can also obtain a perturbative expansion of T in orders of the interaction potential. In passing we mention that due to this similarity, the second-order term of the Feynman amplitude C is called the Born term. The question arises : can one find an equivalent to the Lippmann-Schwinger equation in quantum field theory. This can indeed be done, as was shown by Bethe and Salpeter (ref. 58). The resulting equation is named after them.

First we shall consider the ladder approximated Bethe-Salpeter equation and subsequently we indicate how this result can be generalized.

In the ladder approximation to the Feynman amplitude we take only the contributions of the graphs into account that are shown in figure 12.

$$C^L(k_1 \rightarrow k_2) = C_2(k_1 \rightarrow k_2) + C_{ho}(k_1 \rightarrow k_2) \quad (47)$$

figure 12: the ladder approximated Bethe-Salpeter equation

We discuss the problem in the cm-frame and we drop the spin indices for simplicity. As shown in the figure we can obviously write the ladder approximated Feynman amplitude $C^L(k_1 \rightarrow k_2)$ as the sum of the Born term and the higher order contribution $C_{ho}(k_1 \rightarrow k_2)$. The series of diagrams is infinite. From this observation and from the rules for the Feynman graphs (table 6), we find by extending the considerations that led to the result (45) for the box diagram :

$$C_{ho}(k_1 \rightarrow k_2) = \frac{i}{(2\pi)^4} \int dk C^L(k_1 \rightarrow k) \cdot (M - \gamma^\mu k_\mu - i\eta)^{-1} \cdot (M + \gamma^\mu k_\mu - i\eta)^{-1} C_2(k \rightarrow k_2) \quad (48)$$

From the relations (16) and (19) we know that $(M - \gamma^\mu k_\mu - i\eta)^{-1}$ and $(M + \gamma^\mu k_\mu - i\eta)^{-1}$ are the free field Green functions for each of the nucleons in momentum space. Therefore we can interpret :

$$G(k) = \frac{i}{(2\pi)^4} (M - \gamma^\mu k_\mu - i\eta)^{-1} (M + \gamma^\mu k_\mu - i\eta)^{-1} \quad (49)$$

as the free field Green function for two nucleons in momentum space. The factor $\frac{i}{(2\pi)^4}$ is chosen by convention to simplify its form in coordinate space. Using equations (47), (48) and (49), we see that :

$$C^L(k_1 \rightarrow k_2) = C_2(k_1 \rightarrow k_2) + \int dk C^L(k_1 \rightarrow k) G(k) C_2(k \rightarrow k_2) \quad (50)$$

The expression (50) clearly is a field theoretical analogue of the Lippmann-Schwinger equation.

Formally the extension to the general case in which all graphs are taken into account is performed by

replacing the Born term in (50) by the sum of the contributions from all so-called irreducible graphs. A graph is called irreducible if it is not possible to cut it horizontally through the two nucleon lines and through no meson line, into two connected graphs. By connected we mean that in each of the two pieces, the nucleon lines should be connected by a meson line. For instance the graphs drawn in the figures 10 and 11 are irreducible. The box-graph 9 is reducible.

The general Bethe-Salpeter equation in the cm-system is :

$$C(k_1 \rightarrow k_2) = K(k_1 \rightarrow k_2) + \int dk K(k_1 \rightarrow k) G(k) C(k \rightarrow k_2) \quad (51)$$

$$K(k_1 \rightarrow k_2) = \sum_{\text{irred. graphs}} K_{ig}(k_1 \rightarrow k_2) \quad (52)$$

Evidently the number of irreducible graphs is infinite, therefore in its full generality, the Bethe-Salpeter equation is not very useful for practical problems. Important in this respect are approximations.

5) Lagrangian Field Theory : Conclusions

In the preceding paragraphs, we outlined the more important features of the Lagrangian formulation of quantum field theory in relation to the nucleon-nucleon scattering problem. The basic ingredients of calculations in this formalism are :

- 1) the postulation of an interaction Lagrangian
- 2) the use of perturbation theory

The form of the interaction Lagrangian is restricted by the strong interaction invariances only. Otherwise the choice for a particular form is quite arbitrary. Usually one takes simplicity as a guiding principle.

We mentioned that for some forms of the interaction Lagrangian the theory is renormalizable, while for others, this is not so. For this reason one may assume that renormalizability puts further restrictions on the possible forms of the interaction Lagrangian. However renormalization is important within the frame-work of perturbation theory only. For this reason it is doubtful whether one can attribute much fundamental significance to the renormalizability (ref.56).

Due to practical limitations, we can take only a few of the lowest terms of the perturbative expansion into account. In strong-interaction theory, we deal with coupling constants that are bigger than unity. Therefore there is no reason to expect that the lowest order terms are the most important nor that the perturbative expansion is convergent. It is therefore questionable if one is allowed to use the theory in the case of strongly interacting particles.

There are however, also a few considerations in favour of the formalism for strongly interacting particles. The higher order terms that might give rise to problems

represent the exchange of relatively large masses.

From the simple range-mass relation (2), we notice that these are expected to be important at relatively short distances only. Therefore if we are interested in the interaction at not too short distances, the perturbative approach can be justified. Furthermore due to the exchange of some type of meson, the interaction may be sufficiently repulsive at short distances to prevent the higher order contributions from becoming important (ref. 4). In general we may hope that cancellations occur among the contributions from the omitted higher order diagrams.

6) The Scattering Matrix

The problems involved in Lagrangian field theory in general and in its application to strong interactions in particular, have led to the search for different formulations of quantum field theory. In these new formulations one tries to avoid : 1) making detailed assumptions about the form of the interaction and 2) the use of perturbation theory.

A useful approach is the pragmatic S-matrix formalism, initiated by Heisenberg (ref.59) for ordinary quantum mechanics. In this approach one does not make any detailed assumptions about the interaction. The basic physical ingredients in the S-matrix theory are the asymptotic states of the particles, long before and long after the interaction takes place. The S-matrix elements are defined as the overlap of the initial and final states. Once the S-matrix elements of a reaction are known, then it is possible to derive observable quantities such as cross-sections, lifetimes, etc. The matrix elements can be calculated if we know the form of the interaction. However if we do not know the details of the interaction, it is possible to derive certain relations between experimentally observable quantities under a number of general assumptions, using the dispersion relation technique. These general assumptions include the unitarity of the S-matrix and the usual strong interaction invariance requirements. Furthermore one makes certain assumptions about the analyticity properties of the scattering amplitude. A justification of these analyticity properties is not so easy to give. A proof of these from the principles of field theory is only possible in a few limited cases. In such proofs one tries to relate these analyticity properties with the principle of causality (ref.60).

The field theoretical description which can be used to understand the dispersion relation is developed essentially by Lehmann, Symanzik and Zimmerman (ref. 61). Their approach, which is generally known as the LSZ-formalism, represents an extension of the S-matrix approach of Heisenberg.

In the following we shall consider the S-matrix and the scattering amplitude in the LSZ-formalism. In section 7 we show how we can find dispersion relations for the scattering amplitude. It will be necessary to make certain assumptions about the analyticity of the scattering amplitude. A convenient formulation of these assumptions is given by the Mandelstam representation, which will be discussed in section 8. For simplicity we consider the unrealistic case of spinless neutral nucleons (ref. 52). We shall indicate what modifications are necessary to introduce spin and iso-spin.

The assumption one makes is that the positive energy states form a complete set. We denote these by: $\{f_\alpha(x)\}$. We suppose that $f_\alpha(x)$ is a wave packet. Then we can adopt the normalization :

$$i \int d^3x f_\alpha^*(x) \frac{\overleftrightarrow{\partial}}{\partial x_0} f_\beta(x) = \delta_{\alpha\beta} \quad (53)$$

It is easy to show that this is preserved in time.

Essential concepts in the S-matrix approach are the asymptotic free states in the infinite past and the infinite future. The analogues in the LSZ-field theory are the so called 'in-field $\phi_i(x)$ ' and the 'out-field $\phi_o(x)$ '. Annihilation and creation operators for these field operators can be introduced by their expansion in $\{f_\alpha(x)\}$:

$$\phi_i(x) = \sum_{\alpha} [a_{\alpha} f_{\alpha}(x) + a_{\alpha}^{\dagger} f_{\alpha}^*(x)] \quad (54a)$$

$$\phi_o(x) = \sum_{\alpha} [b_{\alpha} f_{\alpha}(x) + b_{\alpha}^{\dagger} f_{\alpha}^*(x)] \quad (54b)$$

a_{α} and a_{α}^{\dagger} are respectively annihilation and creation operators for a particle in a state $f_{\alpha}(x)$ of the in-field. For the out-field we have similarly, creation and annihilation operators b_{α}^{\dagger} and b_{α} . From the normalization relation (53) we can conclude :

$$a_{\alpha} = i \int d^3x f_{\alpha}^*(x) \overleftrightarrow{\frac{\partial}{\partial x_0}} \phi_i(x) \quad (55a)$$

$$a_{\alpha}^{\dagger} = i \int d^3x \phi_i(x) \overleftrightarrow{\frac{\partial}{\partial x_0}} f_{\alpha}(x) \quad (55b)$$

$$b_{\alpha} = i \int d^3x f_{\alpha}^*(x) \overleftrightarrow{\frac{\partial}{\partial x_0}} \phi_o(x) \quad (55c)$$

$$b_{\alpha}^{\dagger} = i \int d^3x \phi_o(x) \overleftrightarrow{\frac{\partial}{\partial x_0}} f_{\alpha}(x) \quad (55d)$$

In general the field $\phi(x)$ obeys an equation of the form :

$$(\square - M^2) \phi(x) = j(x) \quad (56)$$

where $j(x)$ is some source function for the particles. Furthermore we assume that the usual commutation relations hold.

The creation and annihilation operators for the field $\phi(x)$ are defined by :

$$\phi_{\alpha}(x_0) = i \int d^3x f_{\alpha}^*(x) \overleftrightarrow{\frac{\partial}{\partial x_0}} \phi(x) \quad (57)$$

$$\phi_{\alpha}^{\dagger}(x_0) = i \int d^3x \phi(x) \overleftrightarrow{\frac{\partial}{\partial x_0}} f_{\alpha}(x)$$

The asymptotic fields $\phi_i(x)$ and $\phi_o(x)$ represent the

field $\phi(x)$ for the infinite past and infinite future in the following way :

$$\lim_{x_0 \rightarrow -\infty} \langle A | \phi_\alpha(x_0) | B \rangle = \langle A | a_\alpha | B \rangle$$

$$\lim_{x_0 \rightarrow \infty} \langle A | \phi_\alpha(x_0) | B \rangle = \langle A | b_\alpha | B \rangle$$
(58)

In (58) $|A\rangle$ and $|B\rangle$ represent Fock space wave functions. The scattering operator S can be introduced by the equations :

$$S^{-1} a_\alpha S = b_\alpha \quad S^{-1} a_\alpha^\dagger S = b_\alpha^\dagger$$
(59)

We adopt the phase convention $S|o\rangle = |o\rangle$. Using this phase convention it is easy to see that :

$$|\{\alpha\}, i\rangle = S |\{\alpha\}, o\rangle$$
(60)

In this expression $|\{\alpha\}, i\rangle$ and $|\{\alpha\}, o\rangle$ are Fock space functions for the in - and the out state respectively. The S-matrix elements have therefore the form :

$$S_{\{\alpha\} \rightarrow \{\beta\}} = \langle \{\beta\}, o | \{\alpha\}, i \rangle = \langle \{\beta\}, o | S | \{\alpha\}, o \rangle$$
(61)

In the following we restrict ourselves to the two-particle case. The S-matrix elements can be reduced using the relations (53) - (58) as follows :

$$S_{\alpha, \beta \rightarrow \alpha', \beta'} = \langle \alpha^1, \beta^1 ; o | \alpha, \beta ; i \rangle = \langle \alpha^1, \beta^1 ; o | a_\alpha^\dagger | \beta \rangle$$

$$= i \int d^3y \langle \alpha', \beta', o | \phi_i(y) | \beta \rangle \overleftrightarrow{\frac{\partial}{\partial y_0}} f_\alpha(y)$$
(62)

$$\begin{aligned}
&= \lim_{y_0 \rightarrow -\infty} i \int d^3 y \langle \alpha', \beta' ; 0 | \phi(y) | \beta \rangle \frac{\overleftrightarrow{\partial}}{\partial y_0} f_\alpha(y) \quad (63) \\
&= \lim_{y_0 \rightarrow \infty} i \int d^3 y \langle \alpha', \beta' ; 0 | \phi(y) | \beta \rangle \frac{\overleftrightarrow{\partial}}{\partial y_0} f_\alpha(y) - \\
&\quad - i \int d^4 y \frac{\partial}{\partial y_0} \{ \langle \alpha', \beta' ; 0 | \phi(y) | \beta \rangle \frac{\overleftrightarrow{\partial}}{\partial y_0} f_\alpha(y) \} \quad (64)
\end{aligned}$$

The first of these two terms is simply :

$$\begin{aligned}
\langle \alpha', \beta' ; 0 | \alpha, \beta ; 0 \rangle &= \delta_{\alpha', \alpha} \delta_{\beta', \beta} + \delta_{\alpha', \beta} \delta_{\alpha, \beta'} \\
&\equiv \delta_{\alpha', \beta'; \alpha, \beta} \quad (65)
\end{aligned}$$

By using a Green identity and the Klein-Gordon equation (56), we find for the second term :

$$i \int d^4 y f_\alpha(y) \langle \alpha', \beta' ; 0 | j(y) | \beta \rangle . \text{ So,}$$

$$S_{\alpha, \beta \rightarrow \alpha', \beta'} = \delta_{\alpha', \beta'; \alpha, \beta} + i \int d^4 y f_\alpha(y) \langle \alpha', \beta' ; 0 | j(y) | \beta \rangle \quad (66)$$

The factor $\langle \alpha', \beta' ; 0 | j(y) | \beta \rangle$ can be reduced further :

$$\begin{aligned}
\langle \alpha', \beta' ; 0 | j(y) | \beta \rangle &= \langle \beta' | b_\alpha j(y) | \beta \rangle = \\
&= \lim_{x_0 \rightarrow \infty} i \int d^3 x f_\alpha^*(x') \frac{\overleftrightarrow{\partial}}{\partial x_0} \langle \beta' | P \{ \phi(x), j(y) \} | \beta \rangle \quad (67)
\end{aligned}$$

As we consider the limit $x_0 \rightarrow \infty$ we can always replace $\phi(x) j(y)$ by $P \{ \phi(x) j(y) \}$ as we did in (67). After performing a partial integration with respect to x_0 , one can show, using similar arguments that led from (62) to (64), that :

$$S_{\alpha, \beta \rightarrow \alpha', \beta'} = \delta_{\alpha', \beta'; \alpha, \beta} - \iint d^4y d^4x f_{\alpha'}^*(x) \langle \beta' | P \{j(x), j(y)\} | \beta \rangle f_{\alpha}(y) \quad (68)$$

Note that in the steps leading from (67) to (68) we have tacitly assumed that :

$$(\Box_x - M^2) P \{\phi(x), j(y)\} = P \{j(x), j(y)\} \quad (69)$$

which strictly speaking is not quite correct. There will appear extra terms due to the occurrence of the time-ordered product. These terms turn out not to influence the discussion in a significant way, so we drop them for simplicity (ref. 62).

Although most of the derivations involved in obtaining (68) only apply in the case of wave packets, we frequently use plane wave solutions :

$$f_k(x) = \left\{ \frac{E_k}{M} (2\pi)^3 \right\}^{-1/2} e^{ikx} \quad (70)$$

$$E_k = k_0 = (M^2 + \vec{k}^2)^{1/2} \quad (71)$$

Substituting (70) into (68) we obtain :

$$S(k'_1 k'_2, k_1 k_2) = \langle k'_1 k'_2 | k_1 k_2 \rangle - \left\{ \frac{E_k E_{k'}}{M^2} (2\pi)^6 \right\}^{-1/2} \cdot \iint d^4y d^4x e^{-ik'_1 x} \langle k'_2 | P \{j(x), j(y)\} | k_2 \rangle e^{ik_1 y} \quad (72)$$

We note that the plane wave solutions are eigenfunctions of the energy momentum operator \mathcal{P} , therefore we can write after a few manipulations :

$$S(k'_1 k'_2, k_1 k_2) = \langle k'_1 k'_2 | k_1 k_2 \rangle - \left\{ \frac{E_k E_{k'}}{M^2} (2\pi)^6 \right\}^{-1/2} \cdot$$

$$\cdot (2\pi)^4 \delta^4(k_1 + k_2 - k'_1 - k'_2) \int d^4x e^{-ik'_1 x} \langle k'_2 | P\{j(x), j(0)\} | k_2 \rangle$$
(73)

The transition amplitude F can be defined by :

$$F(k'_1 k'_2, k_1 k_2) = i \left\{ \frac{E_k E_{k'}}{M^2} (2\pi)^6 \right\}^{1/2} \cdot$$

$$\cdot \int d^4x e^{-ik'_1 x} \langle k'_2 | P\{j(x), j(0)\} | k_2 \rangle$$
(74)

$$S_{fi} = \delta_{fi} + i (2\pi)^4 \delta(K_f - K_i) \left\{ \frac{E_{k_1} E_{k_2} E_{k'_1} E_{k'_2}}{M^4} (2\pi)^6 \right\}^{-1/2} F_{fi}$$
(75)

The choice of the normalization allows for a convenient comparison with the Feynman amplitude.

7) Dispersion Relations

In this section we shall derive dispersion relations for the scattering amplitude. Dispersion relations relate the imaginary part of the scattering amplitude to the real part. This is done by considering the analytic continuation of the scattering amplitude to complex values of the energy variables.

It will turn out that the scattering amplitude as defined by equation (74) is not a very convenient tool for this technique. First we shall demonstrate this for the relatively simple case of forward scattering in which $k'_1 = k_1$ and $k'_2 = k_2$. We continue by introducing the more useful 'causal amplitude'. For the causal amplitude we shall derive dispersion relations for a fixed value of the momentum transfer between the two colliding nucleons.

The time-ordered product appearing in the expression for the scattering amplitude (74) can be written as:

$$P\{j(x), j(o)\} = j(x) j(o) \theta(x_0) + j(o) j(x) \theta(-x_0) \quad (76)$$

In (76) $\theta(x_0)$ is the step function, which can be represented by :

$$\theta(x_0) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dw \frac{e^{iwx_0}}{w - i\eta} \quad (77)$$

η : infinitesimal small positive number

Substitution of (77) and (76) in (74) gives :

$$F = \frac{E_k}{M} \iint dx \, dw \frac{e^{-ik \cdot x}}{w - i\eta} \{ \langle k_2 | j(x) j(o) | k_2 \rangle e^{iwx_0} + \langle k_2 | j(o) j(x) | k_2 \rangle e^{-iwx_0} \} (2\pi)^2 ; \quad dx \equiv d\vec{x} dx_0 \quad (78)$$

We assume the existence of a complete set of positive frequency eigenfunctions of the energy momentum operator $\{|p_n\rangle\}$. Expansion of the amplitude in this set gives :

$$\begin{aligned}
 F &= \frac{E_k}{M} \iint dx \, dw \frac{e^{-ik \cdot x}}{w - i\eta} \sum_{p_n} \{ \langle k_2 | j(x) | p_n \rangle \langle p_n | j(0) | k_2 \rangle e^{iwx_0} + \\
 &\quad + \langle k_2 | j(0) | p_n \rangle \langle p_n | j(x) | k_2 \rangle e^{-iwx_0} \} (2\pi)^2 \\
 F &= \frac{E_k}{M} \iint dx \, dw \frac{e^{-ik \cdot x}}{w - i\eta} \sum_{p_n} |\langle k_2 | j(0) | p_n \rangle|^2 \cdot \\
 &\quad \cdot \{ e^{i(wx_0 - k_2 \cdot x + p_n \cdot x)} + e^{i(wx_0 - k_2 \cdot x + p_n \cdot x)} \} (2\pi)^2
 \end{aligned}
 \tag{79}$$

After performing the integrations we get :

$$\begin{aligned}
 F &= (2\pi)^6 \frac{E_k}{M} \sum_{p_n} |\langle k_2 | j(0) | p_n \rangle|^2 \left\{ \frac{\delta^3(\vec{k}_1 + \vec{k}_2 - \vec{p}_n)}{-E_{k_1} - E_{k_2} + E_{p_n} - i\eta} + \right. \\
 &\quad \left. + \frac{\delta^3(\vec{k}_1 - \vec{k}_2 + \vec{p}_n)}{E_{k_1} - E_{k_2} + E_{p_n} - i\eta} \right\}
 \end{aligned}
 \tag{80}$$

If we consider the complex E_{k_1} -plane, then it follows from this result that we have poles in the upper half plane as well as in the lower half plane. So F can not be continued analytically in either one of these planes.

We define the scattering amplitude in a different way. Consider :

$$P\{j(x), j(0)\} = [j(x), j(0)] \theta(x_0) + j(0) j(x) \tag{81}$$

Substitution of (81) into (74), omitting the term

$j(o) j(x)$, gives the so called retarded causal amplitude A^r . Instead of (81) we could also have written :

$$P\{j(x), j(o)\} = -[j(x), j(o)] \theta(-x_0) + j(x) j(o) \quad (82)$$

Substitution of (82) into (74), omitting the term $j(x) j(o)$ gives the so called advanced causal amplitude A^a .

We consider now the analyticity properties of the retarded and advanced causal amplitudes. Starting with the retarded amplitude, we get, after substitution of the representation of $\theta(x_0)$:

$$A^r = \frac{E_k}{M} \iint dx dw \frac{e^{-ikx}}{w - i\eta} [\langle k_2 | j(x) j(o) | k_2 \rangle e^{iwx_0} - \langle k_2 | j(o) j(x) | k_2 \rangle e^{iwx_0}] (2\pi)^2 \quad (83)$$

If we perform the expansion in $\{|p_n\rangle\}$ for this case, then the result is :

$$A^r = (2\pi)^6 \frac{E_k}{M} \sum_{p_n} |\langle k_2 | j(o) | p_n \rangle|^2 \left\{ \frac{\delta^3(\vec{k}_1 + \vec{k}_2 - \vec{p}_n)}{-E_{k_1} - E_{k_2} + E_{p_n} - i\eta} + \frac{\delta^3(\vec{k}_1 - \vec{k}_2 + \vec{p}_n)}{E_{k_1} - E_{k_2} + E_{p_n} + i\eta} \right\} \quad (84)$$

We see that the only difference with the form (80) is the sign of $i\eta$ in the second term. This is very important because it implies that A^r has no poles in the upper half complex E_{k_1} -plane, so it is possible to continue A^r analytically into the upper half plane.

Similarly for the advanced amplitude A^a we have:

$$A^a = (2\pi)^6 \frac{E_k}{M} \sum_{p_n} |\langle k_2 | j(0) | p_n \rangle|^2 \left\{ \frac{\delta^3(\vec{k}_1 + \vec{k}_2 - \vec{p}_n)}{-E_{k_1} - E_{k_2} + E_{p_n} + i\eta} + \frac{\delta^3(\vec{k}_1 + \vec{k}_2 - \vec{p}_n)}{E_{k_1} - E_{k_2} + E_{p_n} - i\eta} \right\} \quad (85)$$

The advanced amplitude can therefore be continued into the lower half complex E_{k_1} -plane. We can define the causal amplitude A by :

$$\begin{aligned} A &= A^r \quad \text{for} \quad \text{Im } E_{k_1} > 0 \\ &= A^a \quad \text{for} \quad \text{Im } E_{k_1} < 0 \end{aligned} \quad (86)$$

The causal amplitude A is manifestly analytic for non-real values of E_{k_1} .

Next we discuss the more general case in which we assume a fixed non-zero momentum transfer. We extend $E_{k_2} + E_{k'_2}$ to the complex plane, and we consider the fixed momentum transfer $k_1 - k'_1$ (ref. 52).

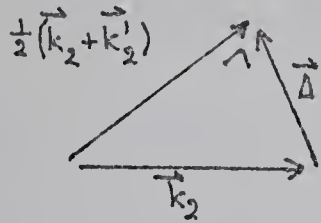
It will be convenient to examine the situation in the Lorentz frame for which $\vec{k}_1 + \vec{k}'_1 = 0$. In this frame the momentum transfer is $2\vec{k}_1 = -2\vec{k}'_1$. We can write :

$$\begin{aligned} k_1 &\equiv \{\vec{\Delta}, E_{\Delta}\} & E_{\Delta} &\equiv (M^2 + \vec{\Delta}^2)^{1/2} \\ k'_1 &\equiv \{-\vec{\Delta}, E_{\Delta}\} & \omega &\equiv E_{k_2} = E_{k'_2} \end{aligned} \quad (87)$$

Using the space-time translation operator, we may write the retarded amplitude in the form :

$$\begin{aligned} A^r &= i \{E_{k_2} E_{k'_2}\}^{1/2} \frac{(2\pi)^3}{M} \int dx \, e^{-\frac{i}{2}x(k_2 + k'_2)} \cdot \\ &\cdot \theta(x_0) \langle k'_1 | [j(\frac{1}{2}x), j(-\frac{1}{2}x)] | k_1 \rangle \end{aligned} \quad (88)$$

From the conservation of energy momentum, one can obtain an expression for $\frac{1}{2}(\vec{k}_2 + \vec{k}'_2)$:



$$|\vec{k}'_2| = |\vec{k}_2| = (\omega^2 - M^2)^{1/2} \quad (89)$$

$$\vec{k}'_2 - \vec{k}_2 = 2\vec{\Delta} \quad (90)$$

From trigonometrics (figure 13)

we see that

$$\frac{1}{2}(\vec{k}_2 + \vec{k}'_2) = (\omega^2 - E_\Delta^2)^{1/2} \vec{e} \quad (91)$$

$$\vec{k}_1 = -\vec{k}'_1 \quad |\vec{e}| = 1 \quad ; \quad \vec{e} \cdot \vec{\Delta} = 0 \quad (92)$$

Substitution of (91) into (68) gives :

$$A^r = i \frac{E_\Delta}{M} (2\pi)^3 \int d\vec{x} e^{-i\vec{x} \cdot \vec{e} (\omega^2 - E_\Delta^2)^{1/2} + i\omega x_0} .$$

$$\cdot \theta(x_0) \langle -\vec{\Delta}, E_\Delta | [j(\frac{1}{2}\vec{x}), j(-\frac{1}{2}\vec{x})] | \vec{\Delta}, E_\Delta \rangle \quad (93)$$

We notice that the x_0 -integration is not only restricted to positive values due to $\theta(x_0)$; but as the commutator $[j(\frac{1}{2}\vec{x}), j(-\frac{1}{2}\vec{x})]$ vanishes for space-like separations, we also have :

$$|x_0| > |\vec{x}| \quad (94)$$

Consider now the analytic continuation of A^r to the complex ω -plane :

$$\omega = \text{Re}\omega + i \text{Im}\omega \quad (95)$$

From the special case of forward scattering, for which $E_\Delta = M$, we know that A^r can not be continued analytically in the lower half ω -plane. The dependence of A^r on ω comes in only via the exponential. We should therefore demand that this be well-behaved. Because of

condition (94), we see that the analytic continuation to the upper half ω -plane is possible if :

$$\text{Im}\omega > \text{Im}(\omega^2 - E_\Delta^2)^{1/2}, \text{ for all values } \omega \quad (96)$$

for which $\text{Im } \omega > 0$

This condition can be satisfied only for negative values of E_Δ^2 , which are unphysical. A possible way-out of this problem was suggested by Bogoliubov (ref. 63). The method consists of replacing E_Δ^2 in the exponential by a negative real constant γ . Then the dispersion relations are derived, and finally the analytic continuation is considered for $\gamma = E_\Delta^2$:

$$A^r = i \frac{E_\Delta}{M} (2\pi)^3 \int dx e^{-i\vec{x} \cdot \vec{e}} (\omega^2 - \gamma)^{1/2} + i\omega x_0 \cdot \theta(x_0) \langle -\vec{\Delta}, E_\Delta | [j(\frac{1}{2}x), j(-\frac{1}{2}x)] | \vec{\Delta}, E_\Delta \rangle \quad (97)$$

A similar expression for A^a can be found by adding a minus-sign and replacing $\theta(x_0)$ by $\theta(-x_0)$. By using the relation $\theta(x_0) = \frac{1}{2}(1 + \varepsilon(x_0))$ in which $\varepsilon(x_0)$ is the signfunction, we can find the real and imaginary parts of A^r and A^a . From these we notice that :

$$\text{Im } A^r = -\text{Im } A^a = \frac{1}{2i}(A^r - A^a)$$

$$\text{Re } A^r = \text{Re } A^a \quad (98)$$

The definition (86) of the causal amplitude A can be extended for non-forward scattering :

$$A \begin{cases} = A^r & \text{for } \text{Im}\omega > 0 \\ = A^a & \text{for } \text{Im}\omega < 0 \end{cases} \quad (99)$$

From the relation (98) and the definition (99), we notice

that the causal amplitude satisfies the so called Schwartz reflection principle :

$$A(\omega^*) = (A(\omega))^* \quad (100)$$

Here we denote explicitly the ω -dependence of A . The relation of this property and the required time-reversal invariance of A can be seen from the expression (97) for A^r , the similar expression for A^a and the definition of A (99). A^r and A^a transform into each other under $\omega \rightarrow -\omega$.

Next we discuss A^r for real values of ω in some detail. Again we assume the existence of a complete set of eigenstates of the energy momentum operator, corresponding to the non-negative values of the energy. In the following, σ^2 denotes the energy of the exchanged particles in their rest frame. We shall use the completeness relation :

$$\sum_{n\mu} \int d\sigma^2 \delta(p_n^2 + \sigma^2) |p_n, \mu\rangle \langle p_n, \mu| = I \quad (101)$$

$|p_n, \mu\rangle$: μ specifies the type and number of intermediate particles with 4-momentum p_n .

In the following we suppress the index μ in $|p_n, \mu\rangle$. The expansion of $\text{Im } A^r$ is :

$$\begin{aligned} \text{Im } A^r &= \frac{E_\Delta}{M} (2\pi)^3 \iint dx \, d\sigma^2 e^{-\vec{x} \cdot \vec{e}} (\omega^2 - \gamma)^{1/2 + i\omega x_0} . \\ \sum_{n\mu} \delta(p_n^2 + \sigma^2) \{ & \langle -\vec{\Delta}, E_\Delta | j(\frac{1}{2}x) | p_n \rangle \langle p_n | j(-\frac{1}{2}x) | \vec{\Delta}, E_\Delta \rangle - \\ & - \langle -\vec{\Delta}, E_\Delta | j(-\frac{1}{2}x) | p_n \rangle \langle p_n | j(\frac{1}{2}x) | \vec{\Delta}, E_\Delta \rangle \} \end{aligned} \quad (102)$$

After performing the space-time integration and a few manipulations, we get :

$$\begin{aligned}
\text{Im } A^r &= (2\pi)^7 \frac{E_\Delta}{M} \int d\sigma^2 \sum_{n\mu} \delta(p_n^2 + \sigma^2) \langle -\vec{\Delta}, E_\Delta | j(0) | p_n \rangle \cdot \\
&\cdot \langle p_n | j(0) | \vec{\Delta}, E_\Delta \rangle \{ \delta^3(\vec{p}_n - (\omega^2 - \gamma)^{1/2} \vec{e}) \delta(\sigma^2 - E_\Delta^2 - \gamma - 2E_\Delta \omega) - \\
&- \delta^3(\vec{p}_n + (\omega^2 - \gamma)^{1/2} \vec{e}) \delta(\sigma^2 - E_\Delta^2 - \gamma + 2E_\Delta \omega) \}
\end{aligned}
\tag{103}$$

If the mass of an intermediate particle is m , we have a discrete value for $\sigma^2 = m^2$ and a continuum for $\sigma^2 > 4m^2$. From (103) we see that the first term vanishes for $\omega \neq \frac{\sigma^2 - E_\Delta^2 - \gamma}{2E_\Delta}$ and the second for $\omega \neq -\frac{\sigma^2 - E_\Delta^2 - \gamma}{2E_\Delta}$. The possible real values of ω for which $\text{Im } A^r$ may be non-zero, are therefore :

$$\text{two discrete values for } \omega = \pm \frac{m^2 - E_\Delta^2 - \gamma}{2E_\Delta} \tag{104}$$

$$\text{two ranges of values for } \frac{4m^2 - E_\Delta^2 - \gamma}{2E_\Delta} < |\omega| < \infty \tag{105}$$

From the Schwartz reflection principle we notice that we can continue A analytically in the complex ω -plane, which has two poles and two cuts, given by (104) and (105) illustrated in figure 14.

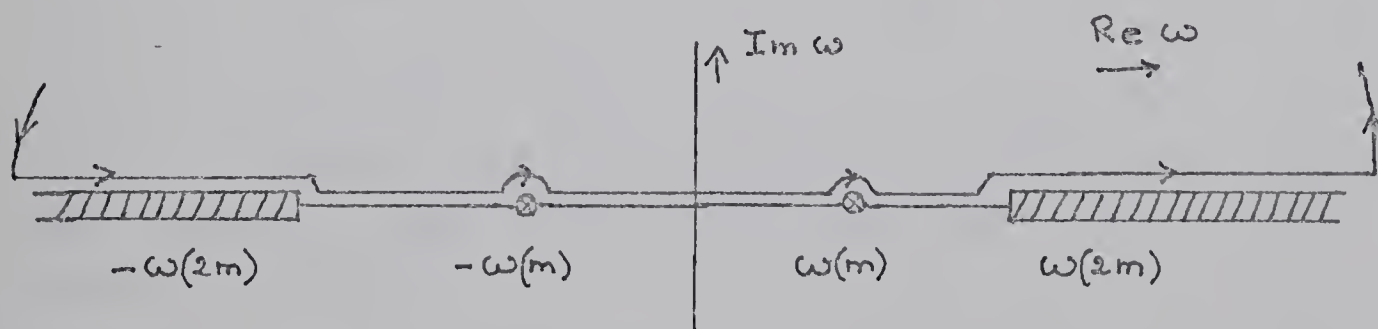


figure 14: pole - and cut structure of the complex ω -plane

We are now able to derive the dispersion relation for A by using the Cauchy theorem. If we assume that $A(\omega)$ vanishes sufficiently rapid for $|\omega| \rightarrow \infty$ then using the contour, illustrated in figure 14 :

$$A(\omega + i\eta) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{A(\omega' + i\eta')}{\omega' - \omega - i\eta} d\omega' \quad ; \quad \eta' < \eta \quad (106)$$

ω , ω' , η and η' are real. η and η' are positive infinitesimal small.

From (100) and (103) - (106) we find dispersion relations of the form :

$$\begin{aligned} A(\omega) = & \frac{g^2}{\omega - \omega(m)} + \frac{g^2}{\omega + \omega(m)} + \frac{1}{\Pi} \int_{-\infty}^{-\omega(2m)} d\omega' \frac{\text{Im } A(\omega')}{\omega' - \omega - i\eta} + \\ & + \frac{1}{\Pi} \int_{\omega(2m)}^{\infty} d\omega' \frac{\text{Im } A(\omega')}{\omega' - \omega - i\eta} \end{aligned} \quad (107)$$

and

$$\begin{aligned} \text{Re } A(\omega) = & \frac{g^2}{\omega - \omega(m)} + \frac{g^2}{\omega + \omega(m)} + \frac{P}{\Pi} \int_{-\infty}^{-\omega(2m)} d\omega' \frac{\text{Im } A(\omega')}{\omega' - \omega} + \\ & + \frac{P}{\Pi} \int_{\omega(2m)}^{\infty} d\omega' \frac{\text{Im } A(\omega')}{\omega' - \omega} \end{aligned} \quad (108)$$

One should keep in mind that these relations were derived for negative values of γ . To find dispersion relations for the physical value $\gamma = E_{\Delta}^2$, we have to consider the analytic continuation of $A(\omega)$ from $\gamma < 0$ to $\gamma = E_{\Delta}^2$. A general proof that this is possible has not yet been given. We shall assume however, that such a continuation is possible.

8) The Mandelstam Representation

As an introduction to the Mandelstam representation we first consider the scattering amplitude which not only applies for the particle-particle scattering, but also for the crossed particle-antiparticle process. We consider again a fixed momentum transfer $k_1 - k'_1$:

$$\text{Im } A = \frac{(E_{k'_1}, E_{k_1})^{1/2}}{M} (2\pi)^3 \int dx e^{-\frac{i}{2}(k'_2 + k_2)x} \langle k'_1 | [j_1(\frac{x}{2}), j_2(-\frac{x}{2})] | k_1 \rangle \quad (109)$$

In (109) we introduced the source operators j_1 and j_2 . The physical significance of these is (ref. 52) :

$$\langle k'_1 | j_1(\frac{1}{2}x) j_2(-\frac{1}{2}x) | k_1 \rangle : \text{accounts for the nucleon-nucleon scattering} \quad (110)$$

$$\langle k'_1 | j_2(-\frac{1}{2}x) j_1(\frac{1}{2}x) | k_1 \rangle : \text{accounts for the 'crossed' nucleon-antinucleon scattering process} \quad (111)$$

The expansion of $\text{Im } A$ into a complete set of intermediate states gives :

$$\begin{aligned} \text{Im } A = & \frac{(E_{k'_1}, E_{k_1})^{1/2}}{M} (2\pi)^7 \int d\sigma^2 \sum_{n\mu} \delta(p_n^2 + \sigma^2) \cdot \\ & \cdot \{ \delta(p_n - k_2 - k'_1) \langle k'_1 | j_1(0) | p_n, \mu \rangle \langle p_n, \mu | j_2(0) | k_1 \rangle - \\ & - \delta(p_n - k_1 + k'_2) \langle k'_1 | j_2(0) | p_n, \mu \rangle \langle p_n, \mu | j_1(0) | k_1 \rangle \} \end{aligned} \quad (112)$$

We consider the first term, which describes the nucleon-nucleon scattering, in some detail. If we assume that there is no bound state of the two particles, the conservation of energy implies a lower bound on the values

σ^2 of the intermediate state of $4M^2$. A bound state would result in a discrete value of σ^2 lower than $4M^2$, because of the negative binding energy. This first term, denoted by $\text{Im } A_s$, reduces to :

$$\text{Im } A_s = \frac{(E_k, E_k)^{1/2}}{M} (2\pi)^7 \sum_{n\mu} \delta(p_n - k_1 - k_2) \cdot \theta(s - 4M^2) \langle k'_1 | j_1(0) | p_n, \mu \rangle \langle p_n, \mu | j_2(0) | k_1 \rangle \quad (113)$$

In (113) s denotes : $s = -(k_1 + k_2)^2$. If we just assume elastic scattering, μ only stands for both the nucleons. As the value of s increases, inelastic channels might open up.

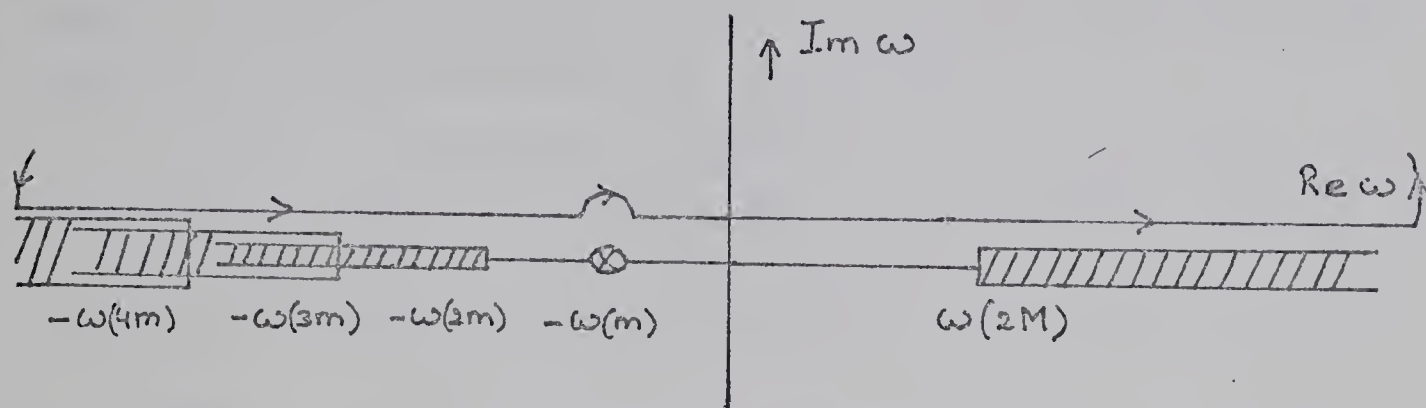
In the second term, for the nucleon-antinucleon scattering, we may interpret $\langle p_n | j_1(0) | k_1 \rangle$ as the factor that denotes the annihilation of the nucleon-antinucleon pair, and the creation of the intermediate mesons. $\langle k'_1 | j_2(0) | p_n \rangle$ is the matrix element for the inversed process. This term, denoted by $\text{Im } A_u$, is expanded as :

$$\begin{aligned} \text{Im } A_u = & - \frac{(E_k, E_k)^{1/2}}{M} (2\pi)^7 \cdot \{ \sum_n \delta(m^2 - u) \langle k'_1 | j_2(0) | p_n; 1 \rangle \langle p_n; 1 | j_1(0) | k_1 \rangle + \\ & + \sum_n \delta(p_n - k_1 + k'_2) \theta(u - 4m^2) \langle k'_1 | j_2(0) | p_n; 2 \rangle \langle p_n; 2 | j_1(0) | k_1 \rangle + \\ & + \sum_n \delta(p_n - k_1 + k'_2) \theta(u - 9m^2) \langle k'_1 | j_2(0) | p_n; 3 \rangle \langle p_n; 3 | j_1(0) | k_1 \rangle \dots \} \end{aligned} \quad (114)$$

there u denotes : $u = -(k_1 - k'_2)^2$

The first term in (114) gives the one particle exchange contribution, the second the two particle exchange etc. (for simplicity we assume one type of meson).

If we define ω as in the last section, we find a pole-cut structure of the complex ω -plane as shown in figure 15 :



figure_15: pole - and cut structure of the complex ω -plane

If we also assume that $A(\omega)$ vanishes sufficiently rapid for $|\omega| \rightarrow \infty$, we get the dispersion relation :

$$A(\omega) = \frac{g^2}{\omega + \omega(m)} + \frac{1}{\pi} \int_{-\infty}^{-\omega(2M)} d\omega' \frac{\text{Im } A_u(\omega')}{\omega' - \omega - i\eta} + \frac{1}{\pi} \int_{\omega(2M)}^{\infty} d\omega' \frac{\text{Im } A_s(\omega')}{\omega' - \omega - i\eta} \quad (115)$$

We notice that equation (113) can be identified with the unitary condition for nucleon-nucleon scattering. The dispersion relation (115) implies that the scattering amplitude for the nucleon-nucleon scattering depends on the singularities due to processes in particle-antiparticle scattering.

One may argue that the properties of the scattering amplitude are dominated by nearby singularities (ref. 64), when we consider the analytic continuation of the scattering amplitude into the complex plane of the appropriate variables. For the case of nucleon-nucleon scattering, this means that we can expect the largest contributions from the one meson exchange, the next important contributions come from the two meson exchange etc.

From (87) one sees that the transformation $k \rightarrow -k$; $k' \rightarrow -k'$ implies that $\omega \rightarrow -\omega$. In that case we are in the 'physical region' of the nucleon-antinucleon process. Similarly equation (114) represents the unitarity for the nucleon-antinucleon scattering.

For a systematic discussion of the crossing relations it is useful to work with the Mandelstam variables s , u and $t = -(k_1 - k'_1)^2$. The dispersion relation (115) in terms of Mandelstam variables is found by using :

$$\omega = \frac{s - 2E_\Delta^2}{2E_\Delta} = - \frac{u - 2E_\Delta^2}{2E_\Delta} \quad ; \quad \Delta^2 = -4t \quad ; \quad s + t + u = 4M^2$$

(116)

then :

$$A(s, u; t) = \frac{g^2}{m^2 - u} + \frac{1}{\pi} \int_{4M^2}^{\infty} ds' \frac{\text{Im } A_s(s', t)}{s' - s - i\eta} +$$

$$+ \frac{1}{\pi} \int_{4m^2}^{\infty} du' \frac{\text{Im } A_u(u', t)}{u' - u - i\eta}$$

(117)

This is the scattering amplitude for a fixed value of t . The physical region for the s -channel scattering is $s \geq 4M^2$, $t \leq 0$, $u \leq 0$. It is easy to see by using the condition $s + t + u = 4M^2$, that this region is equivalently determined by $s \geq 4M^2 - t$ and $t < 0$. By comparison with the first term in the dispersion relation (117), we notice that the integration over s' involves the unphysical region $4M^2 \leq s' \leq 4M^2 - t$. This problem is solved if we assume that the so called Mandelstam representation is correct (ref. 65,66).

The Mandelstam hypothesis states that the amplitude A is an analytic function in the three variables s , t and u except for possible cuts and poles on the real axes of these variables. The three processes that are related to each other correspond to a certain domain of

values for s , t and u . All these processes are described by the same scattering amplitude (see figure 16).

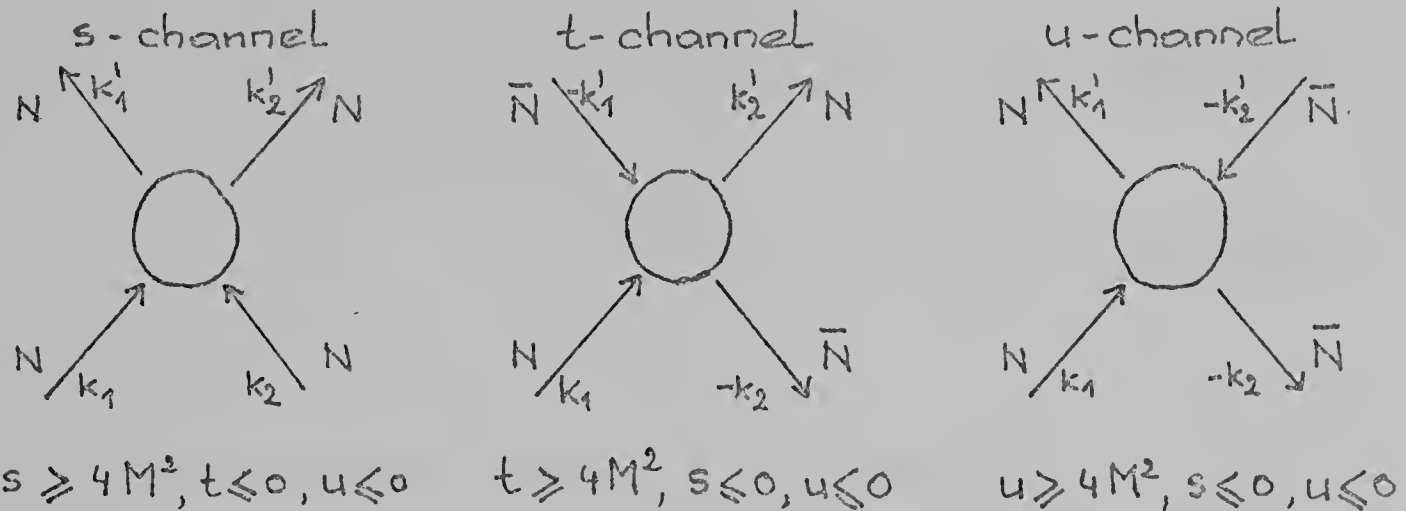


figure 16: the s , t and u channels with their physical domains

Equation (117) is the form of the dispersion relation for a fixed value of t . Similarly for a fixed value of s , the form of the dispersion relation is :

$$\begin{aligned}
 A(t, u; s) = & \frac{g^2}{m^2 - u} + \frac{g^2}{m^2 - t} + \frac{1}{\pi} \int_{4m^2}^{\infty} dt' \frac{A_t(t', s)}{t' - t - i\eta} + \\
 & + \frac{1}{\pi} \int_{4m^2}^{\infty} du' \frac{A_u(u', s)}{u' - u - i\eta}
 \end{aligned} \tag{118}$$

In this expression the integration over the t -variable and the pole in t is due to the unitarity condition for scattering at a fixed value of s in the t -channel.

The dispersion relations (117) and (118) are the so called single spectral representations of the scattering amplitude. The Mandelstam representation is a double spectral representation of the scattering amplitude from which we can deduce the three single spectral representations. The representation exhibits the Mandelstam hypothesis concerning the analyticity of the scattering amplitude :

$$\begin{aligned}
 A(s,t,u) = & \text{pole terms} + \frac{1}{\pi^2} \int_{s_0}^{\infty} ds' \int_{u_0}^{\infty} du' \frac{A(s',u')}{(s'-s)(u'-u)} + \\
 & + \frac{1}{\pi^2} \int_{s_0}^{\infty} ds' \int_{t_0}^{\infty} dt' \frac{A(s',t')}{(s'-s)(t'-t)} + \\
 & + \frac{1}{\pi^2} \int_{u_0}^{\infty} du' \int_{t_0}^{\infty} dt' \frac{A(u',t')}{(u'-u)(t'-t)}
 \end{aligned}
 \tag{119}$$

The double-spectral functions are real and analytic. Clearly singularities appear only for real values of the Mandelstam variables. The three single spectral dispersion relations are easily found by splitting the product in the denominator into a sum.

An attractive graphical illustration of the Mandelstam representation is shown in figure 17 :

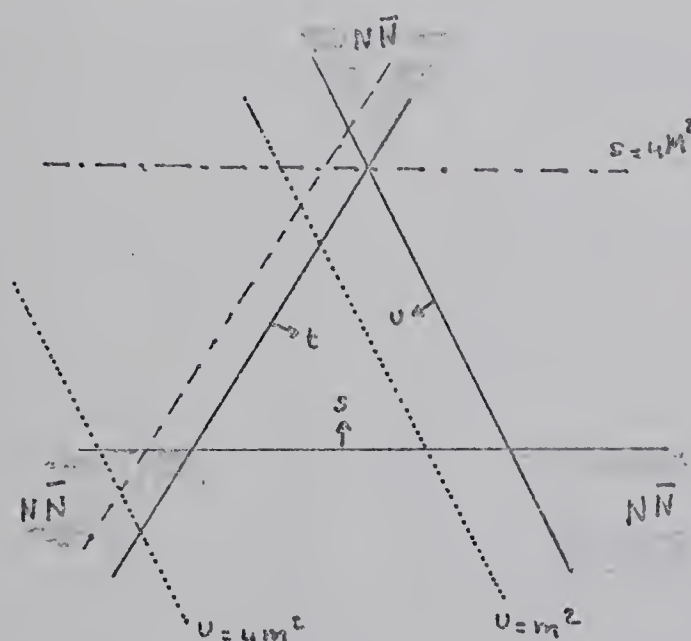
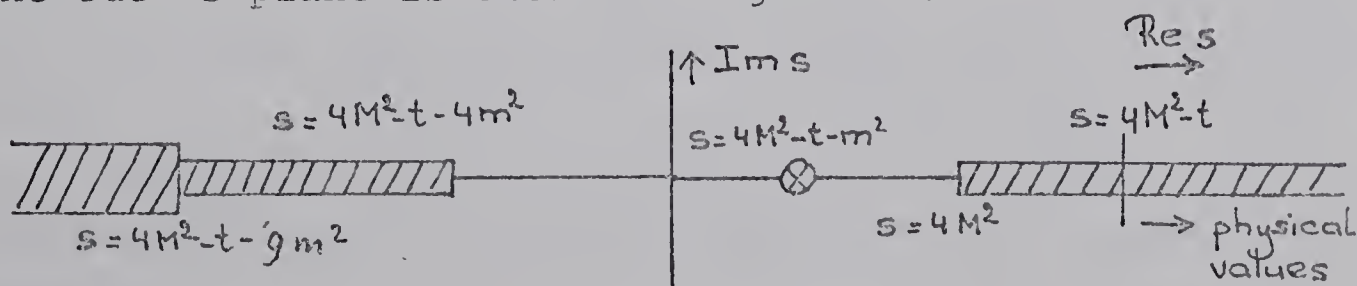


figure 17 :
the pole and cut
structure for the
fixed-t scattering
amplitude is found
by simply looking
at the intersections
of the fixed-t line
with the 'process' lines

The cut s -plane is shown in figure 18.



figure_18:the pole - and cut structure of the complex s-plane

Up to this point we have considered the unrealistic case of neutral spinless nucleons. If we deal with spin 1/2-Fermions instead of Bosons, (74) should be replaced by :

$$F = i \frac{(E_k, E_k)^{1/2}}{M} (2\pi)^3 \int d^3x \bar{u}_{s_1}(k'_1) e^{-ik'_1 x} \cdot \langle s'_2 k'_2 | T\{j_1(x), \bar{j}_2(0)\} | s_2 k_2 \rangle u_{s_1}(k_1) \quad (120)$$

In (120) $j_1(x)$ is the Fermion-source operator :

$$(i\gamma^\mu \frac{\vec{\partial}}{\partial x_\mu} + M) \Psi_1(x) = j_1(x) \quad (121)$$

The Dirac-adjoint source operator $\bar{j}_2(x)$ is :

$$\bar{\Psi}_2(x) (-i\gamma^\mu \frac{\overleftarrow{\partial}}{\partial x_\mu} + M) = \bar{j}_2(x) \quad (122)$$

The rest of the treatment can be given along the same lines as in §7 and §8. Furthermore if we consider the scattering for a given value of the total angular momentum and isotopic spin, we can express the scattering amplitude in terms of an appropriate set of five covariant amplitudes. This is a consequence of the strong interaction invariances and was discussed in some detail in chapter I.*) As there are two values of the isotopic spin, we require a set of ten independent amplitudes. For each of these ten independent amplitudes one can use dispersion relations as shown for the neutral spinless case. The situation is however not as complicated as it seems to be. Due to the Pauli exclusion principle, there are relations among the $3 \times 10 = 30$ double-spectral functions $A(x, y)$. It is important to make a clever choice of the ten independent scattering amplitudes to exhibit the relations implied by the Pauli principle in a convenient fashion.

*) for the non-relativistic case

9) Dispersion Relations : Conclusions

General rigorous derivations of dispersion relations from an axiomatic field theory are not available. We notice that we can prove them for the forward scattering case (no unphysical region (figure 17) or equivalently $\gamma=0$ in (103)). If we adopt the Mandelstam hypothesis, we can write down dispersion relations. These give a new approximation technique, which is not troubled by the deficiencies of the perturbation expansion in Lagrangian field theory. The residues of the pole terms provide a natural operational definition of the coupling-constants. The dispersion relations permit one to discover a number of relations between by crossing connected processes. We can, for instance, use empirical data concerning meson-nucleon scattering to calculate cross-sections for nucleon-nucleon scattering. This comes about, because the meson-nucleon scattering process is by crossing connected with the 'two meson \leftrightarrow nucleon-antinucleon' process. This last reaction accounts for the two meson contribution in nucleon-nucleon scattering.

A disadvantage of the technique is that it only applies for a fixed momentum transfer. This restriction implies that we can not use it for the determination of the off-shell behaviour of the interaction.

Although we need not to make assumptions about the details of the interaction, we should now accept the Mandelstam representation. Another assumption that was used in writing down the dispersion relation was that $A(\omega)$ vanished sufficiently rapidly for $|\omega| \rightarrow \infty$. More precisely this condition is (for the complex s -plane and a fixed value of t) :

$$\lim_{|s| \rightarrow \infty} |A(s, u; t)| < c |s|^{-b} \quad (123)$$

c, b are constants , $b > 0$

If

$$\lim_{|s| \rightarrow \infty} |A(s, u; t)| < c |s|^{+N-b}, \text{ where} \quad (124)$$

N is an integer,

then we can obviously derive the dispersion relation for the function :

$$A_N(s, u; t) = A(s, u; t) \cdot \left\{ \prod_i^N (s - a_i) \right\}^{-1} \quad (125)$$

a_i are real distinct constants.

If we then substitute (125) into the dispersion relation for $A_N(s, u; t)$, we find the relation for $A(s, u; t)$. Written in the variable s , we obtain :

$$\begin{aligned} A(s, t) = & \sum_j \frac{g_j^2}{s - s_j} + \{A_1 s + A_2 s^2 + \dots + A_N s^N\} + \\ & + \frac{1}{\pi} \prod_i^N (s - a_i) \int_{\text{cuts}} ds' \left\{ \prod_i^N (s' - a_i) \right\}^{-1} \text{Im } A(s', t) (s' - s - i\epsilon) \end{aligned} \quad (126)$$

This is the so-called N -subtracted dispersion relation. s_j are the poles for $A(s, u; t)$ in the s - and u -channel. We notice the appearance of N adjustable constants. From field theoretical considerations, it can be shown that no more than two subtractions are necessary (ref. 60).

Appendix Helicity States and Partial Waves

In the elastic nucleon-nucleon scattering, the experimental phase parameters are expressed in the JLS-representation. The theoretical calculations are most conveniently performed by using the helicity representation, introduced by Jacob and Wick (ref. 67). In this section we consider the helicity representation and its relation to the JLS-representation.

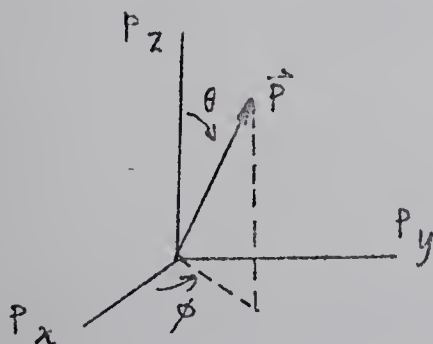
We can specify a two-nucleon system in a given iso-spin state with a cm-momentum \vec{p} and the helicities λ_1 and λ_2 ; by :

$$|\vec{p} \lambda_1 \lambda_2\rangle \quad (1)$$

This representation is very useful for the calculation of Feynman amplitudes. We consider how it is related to a representation in which one specifies the total angular momentum JM ; the magnitude of the cm-momentum p and the helicities λ_1 and λ_2 . To this end we expand $|\vec{p} \lambda_1 \lambda_2\rangle$:

$$|\vec{p} \lambda_1 \lambda_2\rangle = \sum_{JM} \eta_{JM}(\vec{p} \lambda_1 \lambda_2) |pJM \lambda_1 \lambda_2\rangle \quad (2)$$

In the following we shall frequently write \vec{p} in terms of its spherical components $p \theta \phi$, illustrated in the figure. We notice that in the cm-system the total helicity λ is given by :



$$\lambda = \lambda_1 - \lambda_2 \quad (3)$$

If \vec{p} is pointed along the positive z -axis, then :

$$M = \lambda \quad (4)$$

figure 19

so :

$$|p00; \lambda_1 \lambda_2\rangle = \sum_J \eta_J (p00; \lambda_1 \lambda_2) |pJ\lambda\lambda_1 \lambda_2\rangle \quad (5)$$

The state $|p\theta\phi\lambda_1 \lambda_2\rangle$ can be found from (5) by performing a rotation :

$$|p\theta\phi; \lambda_1 \lambda_2\rangle = R(\theta\phi) |p00; \lambda_1 \lambda_2\rangle \quad (6)$$

where $R(\theta\phi) = e^{-iJ_z\phi} e^{-iJ_y\theta} e^{iJ_z\phi}$

using :

$$R(\theta\phi) |JM\rangle = \sum_{M'} \mathcal{D}_{M'M}^J(\phi\theta-\phi) |JM'\rangle \quad (7)$$

and (2), (5), (6), we get the relation :

$$\eta_{JM} (p\theta\phi; \lambda_1 \lambda_2) = \eta_J (p00; \lambda_1 \lambda_2) \mathcal{D}_{\lambda M}^J(\phi\theta-\phi) \quad (8)$$

Suppressing the variables in η_J , we find by substitution of (8) into (2) :

$$|\vec{p}\lambda_1 \lambda_2\rangle = \sum_{JM} \eta_J \mathcal{D}_{\lambda M}^J(\phi\theta-\phi) |pJM\lambda_1 \lambda_2\rangle \quad (9)$$

From the orthonormality relations of the \mathcal{D} -functions follows that for :

$$\eta_J = \left(\frac{2J+1}{4\pi}\right)^{1/2}, \quad (10)$$

the states $\{|pJM\lambda_1 \lambda_2\rangle\}$ are normalized :

$$\langle p'J'M'\lambda'_1 \lambda'_2 | pJM\lambda_1 \lambda_2 \rangle = \delta_{p'p} \delta_{J'J} \delta_{M'M} \delta_{\lambda'_1 \lambda_1} \delta_{\lambda'_2 \lambda_2} \quad (11)$$

The relation of the representation $\{|pJM\lambda_1 \lambda_2\rangle\}$ and

$\{|p_{JMLS}\rangle\}$ is :

$$|p_{JMLS}\rangle = \sum_{\lambda_1 \lambda_2} \left(\frac{2L+1}{2J+1}\right)^{1/2} \left(\frac{1}{2}\lambda_1 \frac{1}{2}\lambda_2 |S\lambda\rangle (L0S\lambda |J\lambda)\right) |p_{JM\lambda_1 \lambda_2}\rangle \quad (12)$$

A proof for this relation can be found in reference 68.

Let us consider matrix elements of a transition operator M in terms of the helicity representation :

$$\{\langle \vec{p}' \lambda'_1 \lambda'_2 | M | \vec{p} \lambda_1 \lambda_2 \rangle\} \quad (13)$$

and see how the matrix elements of M in terms of the representation $\{|p_{JM\lambda_1 \lambda_2}\rangle\}$ can be found from these. Without loss of generality we can choose \vec{p} along the positive z -axis and \vec{p}' in the x - z -plane. Then :

$$|\vec{p} \lambda_1 \lambda_2\rangle = \sum_J \left(\frac{2J+1}{4\pi}\right)^{1/2} |p_{J\lambda_1 \lambda_2}\rangle \quad (14)$$

and

$$|\vec{p}' \lambda'_1 \lambda'_2\rangle = \sum_{J'M'} \left(\frac{2J'+1}{4\pi}\right)^{1/2} d_{M'\lambda'}^{J'}(\theta) |p' J' M' \lambda'_1 \lambda'_2\rangle \quad (15)$$

θ denotes the angle between \vec{p} and \vec{p}'

Upon substitution of (14) and (15) into (13), we get :

$$\langle \vec{p}' \lambda'_1 \lambda'_2 | M | \vec{p} \lambda_1 \lambda_2 \rangle = \sum_J \left(\frac{2J+1}{4\pi}\right) d_{\lambda\lambda'}^J(\theta) \langle p' J \lambda \lambda'_1 \lambda'_2 | M | p_{J\lambda_1 \lambda_2} \rangle \quad (16)$$

*) Brown and Jackson's equation (ref. 4) is wrong, as it includes a summation over M and M -dependencies in the elements.

For the reduced \mathcal{D} -matrix elements $d_{\lambda\lambda'}^J(\theta)$ simple expressions exist in terms of the familiar Legendre polynomials $P_J(\cos\theta)$.

In chapter I we showed in the LSJ-representation that there are five independent scattering matrix elements for each value of the energy and total angular momentum quantum numbers. We have in general one spin-singlet element, one for the uncoupled spin-triplet state and three for the coupled spin-triplet states. This holds for $L \neq 0$ and on-shell scattering, and is a consequence of time-reversal invariance, parity conservation, charge independence and the Pauli principle. The last three principles imply the conservation of total spin. These conservation laws imply the following for matrix elements in the $\{|pJM\lambda_1\lambda_2\rangle\}$ - representation :

time-reversal invariance :

$$\langle \lambda_1' \lambda_2' | M^J(p) | \lambda_1 \lambda_2 \rangle = \langle \lambda_1 \lambda_2 | M^J(p) | \lambda_1' \lambda_2' \rangle$$

$$\text{parity conservation} \quad : = \langle -\lambda_1' -\lambda_2' | M^J(p) | -\lambda_1 -\lambda_2 \rangle$$

$$\text{total spin conservation} \quad : = \langle \lambda_2' \lambda_1' | M^J(p) | \lambda_2 \lambda_1 \rangle \quad (17)$$

In (17) we have used the notation $\langle pJ|M|Jp \rangle \equiv M^J(p)$.

One can also define spin-singlet, 'uncoupled' spin-triplet and three 'coupled' spin-triplet matrix elements in the $\{|pJM\lambda_1\lambda_2\rangle\}$ - representation. To this end we have to combine states to ones with a definite parity and total spin. We obtain :

$$\begin{array}{ll}
\text{spin-singlet} & \frac{1}{\sqrt{2}} (|++>-|-->) \\
& \text{notation :} \\
& \text{pJ dependences are suppressed ,} \\
\text{spin-triplet} & \left\{ \begin{array}{l} \frac{1}{\sqrt{2}} (|+->-| -+>) \\ \frac{1}{\sqrt{2}} (|++>+|-->) \\ \frac{1}{\sqrt{2}} (|+->+| -+>) \end{array} \right. \quad \begin{array}{l} +\equiv +\frac{1}{2} ; -\equiv -\frac{1}{2} \end{array}
\end{array} \quad (18)$$

The spin-singlet matrix element is clearly :

$$M_0^J(p) = \langle ++ | M^J(p) | ++ \rangle - \langle ++ | M^J(p) | -- \rangle \quad (19)$$

The uncoupled spin-triplet matrix element may be found by checking which of the spin-triplet states in (18) does not couple to one of the others. Using the relations (17), it is easy to see that this is the spin-triplet state with the minus-sign. So :

$$\text{'uncoupled' spin-triplet} \quad M_1^J(p) = \langle +- | M^J(p) | +- \rangle - \langle +- | M^J(p) | -+ \rangle \quad (20)$$

$$\text{'coupled' spin-triplet} \quad \left\{ \begin{array}{l} M_{11}^J(p) = \langle ++ | M^J(p) | ++ \rangle + \langle ++ | M^J(p) | -- \rangle \\ M_{22}^J(p) = \langle +- | M^J(p) | +- \rangle + \langle +- | M^J(p) | -+ \rangle \\ M_{12}^J(p) = 2 \langle ++ | M^J(p) | +- \rangle \end{array} \right. \quad (21)$$

$M_{12}^J(p)$ is the non-diagonal element. Clearly the matrix elements for the uncoupled states are the same as the equivalent ones in the JLS-representation. Using the notation of chapter I, we have :

$$M_J(p) = M_0^J(p) \quad (22)$$

$$M_{JJ}(p) = M_1^J(p) \quad (23)$$

Although the functions $\frac{1}{\sqrt{2}}(|+->+|--+>)$ and $\frac{1}{\sqrt{2}}(|++>+|--->)$ span the same subspace as $|L=J+1, S=1>$ and $|L=J-1, S=1>$, there is no reason to believe that these represent similar states. In order to relate the coupled spin-triplet elements, we have to use equation (12) :

$$M_{J+1,J}^J(p) = \frac{1}{2J+1} [(J+1)M_{11}^J(p) + JM_{22}^J(p) + 2\sqrt{J(J+1)}M_{12}^J(p)] \quad (24)$$

$$M_{J-1,J}^J(p) = \frac{1}{2J+1} [JM_{11}^J(p) + (J+1)M_{22}^J(p) + 2\sqrt{J(J+1)}M_{12}^J(p)] \quad (25)$$

$$M^J(p) = \frac{1}{2J+1} [\sqrt{J(J+1)}(M_{22}^J(p) - M_{11}^J(p)) - JM_{12}^J(p)] \quad (26)$$

The representation $\{|pJM\lambda_{12}\rangle\}$ is very useful in itself, but in the context of the low energy nucleon-nucleon scattering it merely serves as an intermediate between the 'theoretical' helicity representation and the 'experimental' JLS-representation. The calculation of the matrix elements in the $\{|pJM\lambda_{12}\rangle$ - representation from those in the helicity representation, is frequently done by introducing a set of so called helicity amplitudes :

$$f_1(z) = \langle \theta' \phi' ; ++ | M(p) | \theta \phi ; ++ \rangle - \langle \theta' \phi' ; ++ | M(p) | \theta \phi ; -- \rangle$$

$$f_2(z) = \langle \theta' \phi' ; ++ | M(p) | \theta \phi ; ++ \rangle + \langle \theta' \phi' ; ++ | M(p) | \theta \phi ; -- \rangle$$

$$f_3(z) = \langle \theta' \phi' ; +- | M(p) | \theta \phi ; +- \rangle (1+z)^{-1} - \langle \theta' \phi' ; +- | M(p) | \theta \phi ; -+ \rangle (1-z)^{-1}$$

$$f_4(z) = \langle \theta' \phi' ; +- | M(p) | \theta \phi ; +- \rangle (1+z)^{-1} + \langle \theta' \phi' ; +- | M(p) | \theta \phi ; -+ \rangle (1-z)^{-1}$$

$$f_5(z) = \langle \theta' \phi' ; ++ | M(p) | \theta \phi ; +- \rangle 2(1-z^2)^{-1/2}$$

(27)

Using the relation (16) and the orthonormality of the Legendre functions, it is easy to express the elements $\{M_i^J(p)\}$ in terms of the helicity amplitudes :

$$M_0^J(p) = \int_{-1}^1 dz P_J(p) f_1(z) \quad (28a)$$

$$M_1^J(p) = \int_{-1}^1 dz \{f_4(z) + f_3(z) \frac{JP_{J+1}(z) + (J+1)P_{J-1}(z)}{2J+1}\} \quad (28b)$$

$$M_{11}^J(p) = \int_{-1}^1 dz f_2(z) P_J(z) \quad (28c)$$

$$M_{22}^J(p) = \int_{-1}^1 dz \{f_3(z) P_J(z) + f_4(z) \frac{JP_{J+1}(z) + (J+1)P_{J-1}(z)}{2J-1}\} \quad (28d)$$

$$M_{12}^J(p) = \int_{-1}^1 dz f_5(z) \left\{ \frac{J(J+1)}{2J+1} \right\}^{1/2} (P_{J+1}(z) - P_{J-1}(z)) \quad (28e)$$

Due to the way the $\{f_i(z)\}$ was defined, no z -dependence other than from $f_i(z)$ and $P_i(z)$ enters these relations. From the definitions of the helicity amplitudes, it is clear that their meaning depends on the scattering amplitude M . In the literature one encounters different helicity amplitudes according to different definitions of M . In the textbook of Brown-Jackson, the helicity amplitudes are defined with respect to the Feynman amplitude. In the papers by Goldberger et. al. (ref. 103) and Wong (ref. 48), these are defined with respect to the scattering amplitude, and a factor $\frac{1}{E}$ is taken out for their f_1, f_2, f_3 and f_4 and a factor $\frac{1}{M}$ for their f_5 . The helicity amplitudes of Brown-Jackson's f_i^{BJ} and Goldberger-Wong's f_i^{GW} are therefore related by :

$$\begin{aligned} \text{for } i : 1, 2, 3, 4 \quad \frac{1}{E} f_i^{GW} &\equiv \frac{M^2}{4\pi E} f_i^{BJ} \\ \text{for } i : 5 \quad \frac{1}{M} f_i^{GW} &\equiv \frac{M^2}{4\pi E} f_i^{BJ} \end{aligned} \quad (29)$$

Furthermore those of Brown-Jackson are more general, as they can be used for off-shell scattering as well. For this reason they also define a sixth helicity amplitude :

$$f_6(z) = \langle \vec{p}'_{-+} | M | \vec{p}_{++} \rangle 2(1-z)^{-1/2} \quad (30)$$

We notice that for on-energy shell, the elements $f_6(z)$ and $f_5(z)$ are equal due to time-reversal invariance.

CHAPTER III

ONE BOSON EXCHANGE MODELS

1) Introduction

Historically the first success of meson field theory in the nuclear interaction was the quantitative establishment of the one pion effect (ref. 71). The one pion exchange process provides a good description of the outer region of the interaction ($r > 1.5$ fm) .

Naturally in the exploration of the more inner lying region, the attention turned to the two pion exchange contribution. In spite of the great efforts, one was not successful in describing this region in terms of the exchange of two pions. One of the problems was the existence of a strong spin-orbit force in the triplet odd states, which could not be explained (ref. 72). Furthermore, the calculational problems in determining the two pion exchange are quite considerable and the handling of three and more pion exchange is not even possible with the present day techniques.

A way out of these problems is offered if one assumes that the interaction is not only due to pion exchange, but to the exchange of other mesons as well. It was shown that the strong spin-orbit force in the triplet odd states could be accounted for by assuming the exchange of vector mesons (ref. 73). The existence of zero strangeness mesons other than the pion was predicted from the Sakata model (ref. 74), which is a predecessor of the quark model. In 1961 Hoshizaki et. al. (ref. 75) determined the type of mesons that are required to obtain the phenomenological Hamada potential (a predecessor of the Hamada-Johnston potential). They concluded that a reasonable

representation of the intermediate range ($0.7 \text{ fm} < r < 1.5 \text{ fm}$) of the potential could be obtained, if one assumes the exchange of an iso-scalar and iso-vector vector meson and an iso-scalar scalar meson besides the pion.

Experimental evidence for the existence of mesons other than the pion came in the first half of the sixties, in the form of resonances of two and three pions (ref. 76). From the existence of strong interactions between the pions it becomes clear that a theory in which only uncorrelated pions are considered, is bound to fail. However one does expect some contributions from the exchange of two or more uncorrelated pions, because even the lightest of the mesons other than the pion is about three times as heavy as the pion (table 7, page 145).

In the OBE (One Boson Exchange)-model, we assume that the effect of the exchange of particles in the form of one Boson takes precedence over that in the form of multiple uncorrelated pions, and that the contribution from these uncorrelated pions can be neglected in a good approximation. Another simplification that one usually makes is that the exchanged mesons are stable particles with zero-width. Furthermore one assumes that the effects of nucleon iso-bars and the interference between the various OBE-contributions can be neglected.

Due to the restriction to OBE-processes, the unitarity requirement is violated, because the OBE-amplitude is real. One has to apply a unitarization procedure, which can be considered as an indirect means of partially accounting for the neglected processes.

A direct and easy way to obtain unitarity is to set the OBE-amplitude equal to the K-matrix elements, to the real part of the transition amplitude T , or directly to the phase-shifts. One can justify this procedure if the phase-shifts are small, as is the case for $L > 1$. A brief

discussion of these methods is given in section 2.

Hoshizaki et. al. (ref.70) have calculated potentials that are associated with the exchanged mesons. By using these potentials in conjunction with the Schrödinger equation, one can satisfy the unitarity requirement. In the derivation of the configuration space representation of the OBE-potentials one has to apply a non-relativistic reduction, which is not justified except for the lower energies. This non-relativistic reduction can be circumvented if the calculations are performed in a momentum space representation. A Schrödinger equation is defined in a non-relativistic context only. Relativistic effects can be included partially by using relativistic phase-space elements in a momentum space representation. This was done by Wong (ref. 48). In a fully relativistic formalism we should use the Bethe-Salpeter equation. In the OBE-model the ladder approximation to it is considered. The practical difficulties in solving the ladder approximated Bethe-Salpeter equation are considerable. This is due to the fact that it is an integral equation in four dimensions. After performing the integration over the angles one still has a two dimensional equation to solve. Therefore, in practice one does not consider the Bethe-Salpeter equation, but a three dimensional approximation to it.

A review of OBE-models that are based on a unitarization through a potential in a Schrödinger equation is given in section 3. OBE-potential models which are given in a momentum space representation form the topic of section 4.

A third method to unitarize the OBE-model is to apply partial wave dispersion relations and find solutions for these. Solutions can be found by using the so-called N/D-method. In the OBE-approximation we replace

the left hand side contribution of the fixed t dispersion relation by pole terms. This method is relativistically invariant. A problem that occurs in the dispersion relations for the OBE-model is to satisfy the threshold behaviour. This problem can be solved at the cost of introducing a non-physical pole. A discussion of the unitarization by solving the partial wave dispersion relations is given in section 5.

Another difficulty that one may encounter is that the amplitudes are badly behaved asymptotically. The Born terms of the Feynman amplitude diverge for large values of the momentum transfer. The partial wave amplitudes in the dispersion relation approach diverge for large values of the energy. Due to this behaviour one may encounter non-convergent integrals. This forces one to use cut-off procedures. Ad-hoc phenomenological forms are used as well as forms based on the eikonal approximation to the higher order vertex correction diagrams, or cut-offs suggested by the Regge theory. Due to the centrifugal barrier for $L \geq 1$, the calculation of these phase parameters is rather insensitive to the specific form that is chosen. However for the S-wave, the effect on the phase-shifts is appreciable.

The free parameters that one chooses in the OBE-models are the meson nucleon coupling constants, the meson masses and the cut-off parameters.

A number of concluding remarks are made in section 6.

2) A Few Simple Unitarization Procedures

In this section we discuss briefly a few simple procedures by means of which the unitarization correction can be made. These methods are expected to be justified for small values of the phase-shifts.

The simplest one can do, is to calculate the partial wave amplitude from the Born term of the Feynman amplitude and set these equal to the phase parameters. Another method is to use geometric unitarity. In this method one uses the fact that if the real part of the partial wave transition amplitudes is known, one automatically knows the phase-shift for elastic unitarity. This is illustrated in figure 20, which shows the Argand plot for elastic scattering.

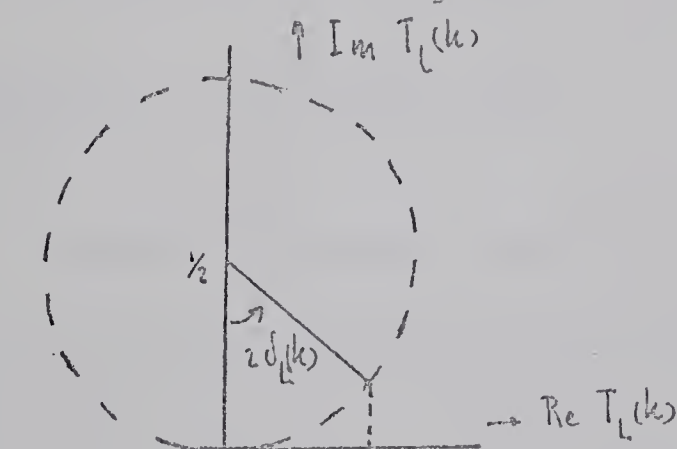


figure 20 :
geometric
unitarization

The unitarization correction is made by equating the calculated real amplitude to $\text{Re} T_L(k)$. Of course for the coupled L-states some modifications should be made. A third simple unitarization scheme, the K-matrix method, will be discussed in some more detail below.

If the phase-shifts are small then the various unitarization procedures should make little difference for the outcome. One can not expect that such simple unitarization procedures can be used for the S-waves, as the phase parameters are quite large in this case. Köpp and Krammer (ref. 77) used it to analyze partial waves for

which $J \geq 2$ only. Somewhat bolder is the OBE-model of Sawada, Ueda, Watari and Yonezawa, collectively known as SUWY (ref. 78). Their OBE-analysis included partial waves for which $L \geq 1$. Their model is quite well-known and we shall discuss it as an example of the application of K-matrix unitarization in OBE-theory.

In the SUWY-model the Feynman amplitude Born term is related to the transition matrix $M(\vec{k}', \vec{k})_{s \rightarrow s'}$, that was discussed in chapter I, sections 3 and 4. We remember that the elements $M(\vec{k}', \vec{k})_{s \rightarrow s'}$ are scattering amplitudes. The relation between scattering amplitudes and the Feynman amplitude C is discussed in the appendix attached to this chapter. Using the non-relativistic phase-space factor $(2\pi)^{-3}$, we obtain the relation :

$$M(\vec{k}', \vec{k})_{s \rightarrow s'} = \sum_i \frac{E_k}{4\pi} C_i^B(\vec{k}', \vec{k})_{s \rightarrow s'} \quad (1)$$

where i denotes the meson that is exchanged

The Born term of the Feynman amplitude is known for each exchanged meson that is considered in the model. The way SUWY proceeded was to calculate the elements $M(\vec{k}', \vec{k})_{s \rightarrow s'}$ for the spin state representation $\{|S m_s\rangle\}$. A convenient way to do this is to use the helicity state representation. The relation between spin space matrix elements of a transition matrix in the $|S m_s\rangle$ - representation and the helicity amplitudes $\{f_i\}$ is given explicitly in reference 48. If we calculate the helicity amplitudes for the Born term of the Feynman amplitude, one can easily find $\{\langle S m'_s | M(k', k) | S m_s \rangle\}$ from these relations. After one has obtained these matrix elements one can use the formalism written down in chapter I, section 4, to calculate the transition matrix elements T_J^B , T_{JJ}^B , $T_{J \pm 1, J}^B$ and T_B^J .

SUWY essentially followed this method. A more direct way to obtain these results is to calculate the helicity amplitudes directly for the transition operator T^B , and then use (22) - (26) of the appendix attached to chapter II. The helicity amplitudes for T^B are easily found from (1) and the relation :

$$M(\vec{k}', \vec{k}) = \frac{4\pi}{2ik} T(\vec{k}', \vec{k}) \quad (2)$$

The elements T_J^B , T_{JJ}^B , $T_{J\pm 1, J}^B$ and T_B^J are real, therefore the unitarity conditions (chapter I, (35) and (36)) can not be satisfied. The procedure employed by SUWY is to identify the OBE T-matrix with the K-matrix. The important property of the K-matrix is that its elements are real. The K-operator is related to the S-operator by :

$$S = (1 + \frac{i}{2}K) (1 - \frac{i}{2}K)^{-1} \quad (3)$$

For the spin-singlet state the S-matrix element is :

$$S_J = e^{2i\delta_J^1} = (1 + i \tan \delta_J^1) (1 - i \tan \delta_J^1)^{-1} \quad (4)$$

We can therefore identify $K_J = 2 \tan \delta_J^1$. Similarly for the uncoupled spin-triplet state $K_{JJ} = 2 \tan \delta_J^3$. For the two coupled spin-triplet states the relation between S-matrix and K-matrix elements is a bit more involved, but can be found straight forwardly. The result is :

$$\begin{pmatrix} S_{J-1, J} & S_J \\ S_J & S_{J+1, J} \end{pmatrix} = \quad (5a)$$

$$D^{-1} \begin{pmatrix} 1 + (\frac{K_J}{2})^2 - \frac{1}{4} K_{J+1, J} K_{J-1, J} - \frac{1}{2} (K_{J+1, J}^{-K_{J-1, J}}) & K_J \\ K_J & 1 + (\frac{K_J}{2})^2 - \frac{1}{4} K_{J+1, J} K_{J-1, J} + \frac{1}{2} (K_{J+1, J}^{-K_{J-1, J}}) \end{pmatrix}$$

$$\text{where } D = 1 - \frac{1}{2}(K_{J+1,J} + K_{J-1,J}) - \frac{1}{4}K_{J+1,J}K_{J-1,J} - \frac{1}{4}(K^J)^2 \quad (5b)$$

The expressions of the S-matrix elements $S_{J-1,J}$, S^J , $S_{J+1,J}$ in terms of the bar phase parameters can be found from chapter I, equation 36. The OBE-model phase parameters follow directly by equating :

$$T^B = K \quad (6)$$

SUWY considered the exchange of an iso-scalar and an iso-vector scalar meson, and an iso-vector pseudo scalar meson with the η -meson mass. Their analysis also included the well-established pion, η -meson, ω -meson and the ρ -meson. The P-wave and D-wave phase parameters were used to determine the parameters of the OBE-contributions. For the $L \geq 3$ partial waves the phase parameters are quite small and largely dominated by the pion exchange. SUWY concluded that for a satisfactory fit the contributions of the iso-vector scalar - and pseudo scalar meson were not necessary. Also the contribution of the η -meson is not required. Leaving these three mesons out of consideration, there are seven adjustable parameters in their model : the mass of the scalar meson and the coupling constants. The predetermined and adjusted values for the masses and coupling constants are given in table 8, page 145. The fit was performed by comparing the calculated phase parameters with those of the phenomenological Hamada-Johnston potential. The quality of the fit of the calculated phase parameters with those from a Yale (ref. 79) and Livermore (ref. 80) phase-shift analysis is illustrated in the figures 21 and 22. We see that a reasonable agreement is obtained.

A review of OBE-models in which K-matrix unitarization is used can be found in an article by Ogawa and the SUWY-authors (ref. 81). In these various OBE-models one

usually restricts the consideration to scalar, pseudo scalar and vector mesons. An important argument in this is that experimentally no low energy pseudo vector mesons or tensor mesons have been found. Furthermore it turned out that the elastic scattering data can be fitted satisfactory without these mesons, although minor improvements of the 3P_0 phase-shift have been reported by including tensor mesons (ref. 82). Agreement exists among the various authors that besides the pion, a scalar meson and the ω -meson are indispensable.

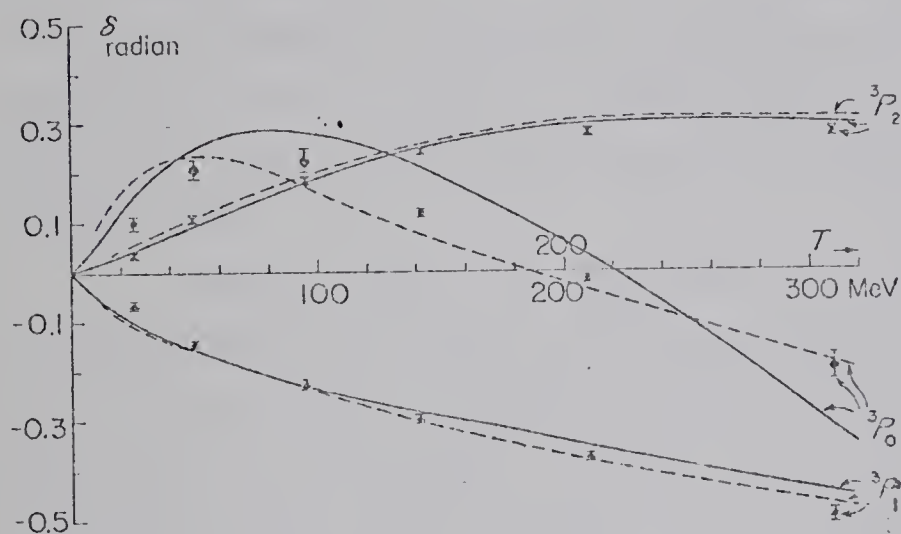


figure 21 :
the SUWY triplet
P-phase-shifts

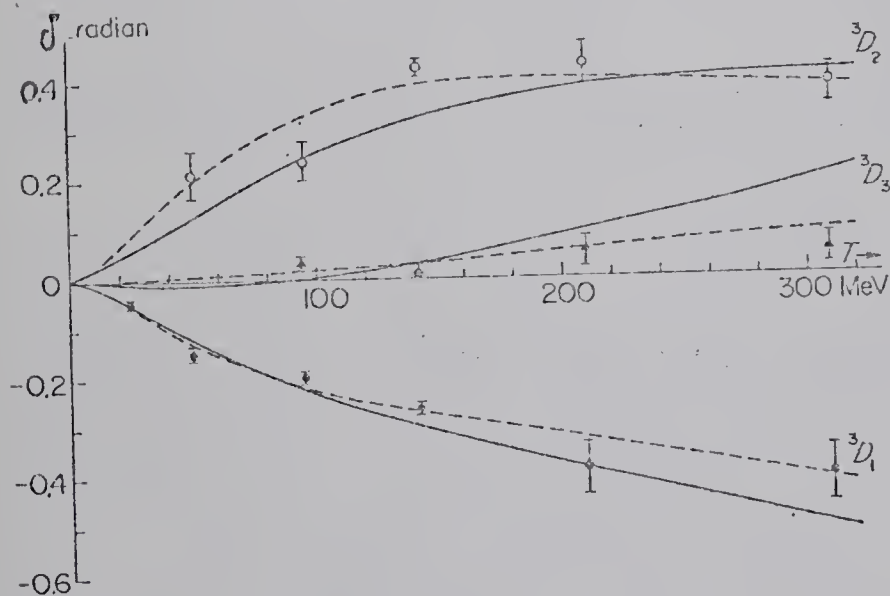


figure 22 :
the SUWY triplet
D-phase-shift

full curve : SUWY phase-shifts
dotted curve : Yale phase-shifts (ref. 79)
experimental points are taken from reference 80

3) One Boson Exchange Potentials *)

Several methods have been designed to associate a potential with an OBE-contribution to the interaction (ref. 31). A simple and frequently used method is to define a potential by interpreting the Born term of the Feynman amplitude as a spin-space matrix element of a potential.

In this section we consider the configuration space OBE-potentials. First we show how expressions for scalar, pseudo scalar and vector mesons are found. As examples of successful OBE-potentials we discuss the models of Bryan and Scott (ref. 83) and Ueda and Green (ref. 84).

Wong (ref. 48) defines the potential in a slightly different manner. He includes relativistic effects partially by using a relativistic phase-space element. We discuss his approach briefly and consider an OBE-potential of this type, constructed by Erkelenz et. al. (ref. 85) as an example. Finally a few concluding remarks about configuration space OBE-potentials are made.

The derivation of the scalar meson potential is discussed in some detail. The same techniques apply for pseudo scalar and vector mesons, and we just state the results. We consider iso-scalar mesons for simplicity. The extension to iso-vector mesons is easily made by replacing the coupling constant g^2 by $\vec{\tau}^1 \cdot \vec{\tau}^2 g^2$.

We consider both the direct and the derivative coupling. Fortunately in the case of a scalar meson, matters are simplified as the Born term for the derivative coupling vanishes. This can be seen by substitution of :

$$-i\partial_\mu \phi = p_\mu \phi \quad ; \quad \vec{p} = \vec{k}' - \vec{k} : \text{momentum transfer} \quad (7)$$

*) In the following we use the convention $a.b. = a_0 b_0 - \vec{a} \cdot \vec{b}$

into the interaction Hamiltonian. The result follows immediately by using Dirac's equation and the on-shell condition $E_k = E_{k'}$.

The Born term for the Feynman amplitude C_B in the case of a scalar meson (C in this derivation differs in a normalization factor $(2\pi)^{-6}$ from the definition in chapter II) is :

$$C_B^S = \frac{M^2}{E_k E_{k'}} \bar{u}_{s'_1}(-k') \sqrt{(4\pi)g_s} u_{s_1}(-k) \{ (k'-k)^2 - m^2 \}^{-1} \cdot \bar{u}_{s'_2}(k') \sqrt{(4\pi)g_s} u_{s_2}(k) \quad (8)$$

k and k' are the momenta in the cm-system before and after the interaction. The basic Dirac spinor $u_s(k)$ is :

$$u_s(k) = \left(\frac{E_k + M}{2M} \right)^{1/2} \begin{pmatrix} 1 \\ \frac{\vec{\sigma} \cdot \vec{k}}{E_k + M} \end{pmatrix} |s> \quad (9)$$

Substitution of (9) in (8) gives :

$$C_B^S = \frac{(E_k + M)(E_{k'} + M)}{4E_k E_{k'}} \langle s'_1 s'_2 | \left(1 - \frac{(\vec{\sigma}^1 \cdot \vec{k})(\vec{\sigma}^1 \cdot \vec{k}')}{(E_k + M)(E_{k'} + M)} \right) \cdot \left(1 - \frac{(\vec{\sigma}^2 \cdot \vec{k})(\vec{\sigma}^2 \cdot \vec{k}')}{(E_k + M)(E_{k'} + M)} \right) | s_1 s_2 \rangle 4\pi g_s^2 \{ (k'-k)^2 - m^2 \}^{-1} \quad (10)$$

The momentum space potential can be defined by :

$$C_B^S = \langle s'_1 s'_2 | V^S(k, k') | s_1 s_2 \rangle \quad (11)$$

In order to make sure that the coordinate representation of the potential has a convenient analytic form, we assume that the non-relativistic reduction can be used (ref. 3) :

$$\frac{E_k + M}{2E_k} \approx 1 - \frac{k^2}{4M^2} \quad (12)$$

It is convenient to introduce the momentum transfer p and the momentum $q = \frac{1}{2}(k' + k)$. Making the reduction (12), we obtain in terms of q and p after a little manipulation :

$$V^S(p, q) = -4\pi g_S^2 \{ \vec{p}^2 + m^2 \}^{-1} \left\{ 1 - \frac{\vec{q}^2}{M^2} + \frac{i(\vec{\sigma}^1 + \vec{\sigma}^2) \cdot (\vec{q} \times \vec{p})}{4M^2} \right\} \quad (13)$$

In (13) we have also replaced $(p^2 - m^2)^{-1}$ by $-(\vec{p}^2 + m^2)^{-1}$. In this replacement we have put $E_{k'} = E_k$. This implies that one assumes that only momentum is transferred by the meson but no energy. This means that the so called meson retardation is ignored in this description.

The coordinate representation of the potential is found by taking the Fourier transform :

$$V(\vec{r}, \vec{q}) = (2\pi)^{-3} \int d^3p V(\vec{p}, \vec{q}) e^{i\vec{p} \cdot \vec{r}} \quad (14)$$

so :

$$-4\pi g_S^2 (\vec{p}^2 + m^2)^{-1} \frac{\vec{q} \cdot \vec{r}}{F \cdot \vec{T}} - g_S^2 \frac{e^{-mr}}{r} \quad r \equiv |\vec{r}| \quad (15)$$

$$4\pi g_S^2 (\vec{p}^2 + m^2)^{-1} \frac{\vec{q}^2}{M^2} \frac{\vec{q} \cdot \vec{r}}{F \cdot \vec{T}} - \frac{1}{2M^2} g_S^2 \left[\frac{e^{-mr}}{r} \Delta + \Delta \frac{e^{-mr}}{r} - \frac{1}{2} \{ \Delta \frac{e^{-mr}}{r} \} \right] \quad (16)$$

$$\text{where } [\Delta \frac{e^{-mr}}{r}] = m^2 \frac{e^{-mr}}{r} - 4\pi \delta(r)$$

$$-\frac{4\pi}{4M^2} g_S^2 (\vec{p}^2 + m^2)^{-1} i(\vec{\sigma}^1 + \vec{\sigma}^2) \cdot (\vec{q} \times \vec{p}) \frac{\vec{q} \cdot \vec{r}}{F \cdot \vec{T}} \frac{g_S^2}{2M^2} \frac{1}{r} \frac{d}{dr} \frac{e^{-mr}}{r} \vec{L} \cdot \vec{S} \quad (17)$$

$$\text{where } \vec{S} = \frac{1}{2}(\vec{\sigma}^1 + \vec{\sigma}^2)$$

The total configuration space scalar potential becomes :

$$V^S = g_S^2 \left[(-1 + \frac{m^2}{4M^2}) \frac{e^{-mr}}{r} - \frac{1}{2M^2} (\Delta \frac{e^{-mr}}{r} + \frac{e^{-mr}}{r} \Delta) + \right. \\ \left. + \frac{1}{2M^2} \frac{1}{r} \frac{d}{dr} \left(\frac{e^{-mr}}{r} \right) \vec{L} \cdot \vec{S} - \frac{4\pi}{4M^2} \delta(r) \right] \quad (18)$$

We notice the appearance of the δ -function, which shows that the potential is not well-behaved at the origin.

The pseudo-scalar derivative coupling is identical to the direct coupling for the Born term and the on-energy shell condition. This can be seen from (7) and the Dirac equation :

$$\mathcal{H}'_p = \sqrt{(4\pi)} \frac{f_p}{m} \sum_{\mu=1}^3 \bar{\Psi} \gamma^5 \gamma^\mu \Psi \frac{\partial}{\partial x_\mu} \phi = \sqrt{(4\pi)} i \frac{2f_p M}{m} \bar{\Psi} \gamma^5 \Psi \phi \quad (19)$$

The total interaction Hamiltonian for the pseudo-scalar becomes :

$$\mathcal{H}_p = \sqrt{(4\pi)} G_p \sum_{\mu=1}^3 \bar{\Psi} i \gamma^5 \Psi \phi \quad ; \quad G_p = g_p + \frac{2f_p M}{m} \quad (20)$$

Proceeding along the same lines as for the scalar meson, we get :

$$\begin{aligned} V^p &= G_p^2 \left[\frac{m}{12M} \frac{e^{-mr}}{r} \vec{\sigma}^1 \cdot \vec{\sigma}^2 + \frac{m^2}{4M^2} \left(\frac{1}{m^2 r^2} + \frac{1}{mr} + \frac{1}{3} \right) \frac{e^{-mr}}{r} S_{12} - \right. \\ &\quad \left. - \frac{4\pi}{12M^2} \delta(r) \vec{\sigma}^1 \cdot \vec{\sigma}^2 \right] \end{aligned} \quad (21)$$

For vector mesons the interaction Hamiltonian :

$$\mathcal{H}^V = (4\pi)^{1/2} \bar{\Psi} \left[g_V \gamma^\mu \phi_\mu + \frac{f_V}{4M} \sigma^{\mu\nu} (\partial_\nu \phi_\mu - \partial_\mu \phi_\nu) \right] \Psi \quad (22)$$

Using (7) :

$$\mathcal{H}^V = (4\pi)^{1/2} \bar{\Psi} \left[g_V \gamma^\mu \phi_\mu + \frac{f_V}{2M} i \sigma^{\mu\nu} (k' - k)_\nu \phi_\mu \right] \Psi \quad (23)$$

A more convenient form can be found after some algebra, by using Dirac's equation (ref. 86) :

$$\mathcal{H}^V = (4\pi)^{1/2} \bar{\Psi} \left[(g_V + f_V) \gamma^\mu \phi_\mu - \frac{f_V}{2M} (k' + k)_\mu \phi_\mu \right] \Psi \quad (24)$$

The vector meson potential, which can be derived from this form is :

$$\begin{aligned}
 V^V = & g_V^2 \frac{e^{-mr}}{r} + (g_V^2 + g_V f_V) \frac{m^2}{4M} \frac{e^{-mr}}{r} - g_V^2 \frac{1}{2M^2} \left(\Delta \frac{e^{-mr}}{r} + \frac{e^{-mr}}{r} \Delta \right) + \\
 & + (g_V + f_V)^2 \frac{m^2}{6M^2} \frac{e^{-mr}}{r} \vec{\sigma}^1 \cdot \vec{\sigma}^2 - (g_V + f_V)^2 \frac{m^2}{4M^2} \left(\frac{1}{3} + \frac{1}{mr} + \frac{1}{m^2 r^2} \right) \cdot \\
 & \cdot \frac{e^{-mr}}{r} S_{12} + (3g_V^2 + 4g_V f_V) \frac{1}{2M^2} \frac{1}{r} \frac{d}{dr} \frac{e^{-mr}}{r} \vec{L} \cdot \vec{S} - (g_V^2 + g_V f_V) \cdot \\
 & \cdot \frac{4\pi}{2M^2} \delta(r) - (g_V + f_V)^2 \frac{4\pi}{6M^2} \delta(r) \vec{\sigma}^1 \cdot \vec{\sigma}^2 \quad (25)
 \end{aligned}$$

The total OBE-potential has the general form :

$$V^{OBE} = \sum_i V^i \quad (26)$$

The summation is performed over the mesons i in the model. The phase parameters can be calculated along the lines of the formalism of chapter I, section 5.

Problems arise due to the singularities at the origin. We notice besides the Dirac δ -function, the occurrence of terms of order r^{-3} .

In the OBE-model of Bryan and Scott (ref. 83), the regularization of the singularities at the origin was accomplished by multiplying the momentum space representation of the potential with the form factor :

$$\Lambda^2 ((k' - k)^2 - \Lambda^2)^{-1} \quad \Lambda : \text{the cut-off mass} \quad (27)$$

To see that this factor provides the desired regularization, we consider the following identity :

$$\Lambda^2 (\vec{p}^2 + \Lambda^2)^{-1} (\vec{p}^2 + m^2)^{-1} = \Lambda^2 (\Lambda^2 - m^2)^{-1} ((\vec{p}^2 + m^2)^{-1} - (\vec{p}^2 + \Lambda^2)^{-1}) \quad (28)$$

In (28) we have replaced $(k'-k)^2$ by $-\vec{p}^2$. From the form (28) it is clear that the effect of the form factor is that each potential term $V_i(m_i)$ in (26) is replaced by :

$$V_i(m_i) \rightarrow \Lambda^2 (\Lambda^2 - m_i^2)^{-1} (V_i(m_i) - V_i(\Lambda)) \quad (29)$$

From inspections of the potentials (18), (21) and (25), it can be verified that in this way the δ -function and the r^{-3} singularity are removed. Only well-behaved terms of order r^{-2} and higher remain. The value of Λ , used by Bryan and Scott, was 1500 MeV.. Due to this large value $V_i(\Lambda)$ falls off rapidly as a function of r .

In their model Bryan and Scott considered besides the pion, the well-established η -meson, ρ -meson and ω -meson. Furthermore they included an iso-vector and an iso-scalar meson. The values of the scalar mesons, coupling constants and the cut-off mass Λ were found by fitting the calculated phase-shifts with those of the phase-shift analysis of Arndt-McGregor (ref. 87). The values for the predetermined and adjusted masses and coupling constants is given in table 9, page 146. The fitting was performed for three values of the laboratory kinetic energy and included all S-, P- and D-waves. A reasonable fit could be obtained except for the 1D_2 -phases, which were predicted too low. This could be expected as none of the potentials (18), (21) and (25) includes a quadratic spin-orbit term. Phenomenological potential analysis indicates that such a term is necessary for a simultaneous fit to the spin-singlet 1S_0 - and 1D_2 -states (ref. 88). An unsatisfactory feature of the Bryan-Scott potential is that the unrealistic large value 17.26 for the ω -meson coupling constant was required. Estimates from electro-magnetic form factors suggest a value of at most 9 (ref. 89). The

figures 24 and 25, on page 122, display the result for the triplet P-waves and the 1D_2 -wave.

Ueda and Green (ref. 84) derived the potential in a slightly different way, based on a formalism developed essentially by Fock in 1934 (ref. 90). A discussion of this formalism is given in reference 91. The types of mesons they include in their model I, are the same as in the model of Bryan and Scott, except that they include another iso-scalar scalar meson. All meson masses were predetermined. One iso-scalar scalar was arbitrarily fixed at $3m_\pi$. The other iso-scalar scalar and the iso-vector scalar mesons were given the masses 1070 MeV. and 1016 MeV. respectively. These were considered as experimentally established values. (It is of interest to note that in the particle data sheets of January 1976 (page 92) in the mass range of 1000 - 1200 MeV. two iso-scalar scalars are listed, the S^* at 993 MeV. and the ϵ at 1200 MeV.. The S^* seems not to couple to non-strange Baryons. An iso-vector scalar, the δ is listed at 960 MeV.). The free parameters in their model are the coupling constants and the cut-off masses. They considered one for the pion ($\Lambda_\pi = 2532.4$ MeV.) and one common to the other mesons ($\Lambda = 1184.3$ MeV.). Values for masses and coupling constants are stated in table 10, page 146. In total they had 10 adjustable parameters whereas Bryan and Scott had 11. The calculated phase parameters were fitted to phase parameters of Livermore (ref. 93) and Yale (ref. 94) for S-, P- and D-waves (fig. 26 on page 123). The fit they obtain is better than the one in Bryan and Scott's model. Although they had also trouble in simultaneously fitting the 1S_0 - and 1D_2 -phase-shifts at higher energies. An ad-hoc inclusion of a quadratic angular momentum term improved the fit to a set of 76 Yale phase parameters from a reduced $\chi^2 = 1.6$ to a reduced $\chi^2 = 1.2$.

Wong's method to calculate the OBE-potentials differs from the one we have discussed, in that he considers potentials which are non-local due to the use of a relativistic phase-factor. We illustrate his method for a spin independent potential and indicate how the generalization is made. The results of an application of this method by Erkelenz et. al. (ref. 85) is briefly considered.

We notice that in applying the Fourier transform (14), we used a non-covariant phase-factor. Wong considers a form for the potential in which the relativistic phase-factor $\frac{M}{E}(2\pi)^{-3}$ is used. As we discussed in chapter I, section 8, this leads to a non-local potential $V(\vec{r}', \vec{r})$ in coordinate space. The potential in momentum space is again put equal to the Born term of the Feynman amplitude, therefore we can write :

$$\frac{M}{E_k} C^B(k', k) = \int d\vec{r}' d\vec{r} e^{-i\vec{k}' \cdot \vec{r}'} V(\vec{r}', \vec{r}) e^{i\vec{k} \cdot \vec{r}} \quad (30)$$

Wong considers the coordinate transformation $\vec{r}, \vec{r}' \rightarrow \vec{r}, \vec{r}' - \vec{r}$. Furthermore instead of \vec{k} and \vec{k}' or \vec{p} and \vec{q} as in (13), he uses \vec{k} and $\vec{p} = \vec{k} - \vec{k}'$. It turns out that the non-locality due to the relativistic phase-space element can be accounted for by using the separable form :

$$V(\vec{r}', \vec{r}) = N(|\vec{r}' - \vec{r}|) U(r, k^2) \quad (31)$$

It is easy to verify this. To this end we consider the Bauer expansion of the integrals in (30). The potential is a scalar in coordinate space, so :

$$\begin{aligned} \frac{M}{E_k} C^B(p, k) &= \left\{ -4\pi \int_0^\infty d|\vec{r} - \vec{r}'| |\vec{r} - \vec{r}'|^2 j_0(k|\vec{r}' - \vec{r}|) N(|\vec{r}' - \vec{r}|) \right\} \cdot \\ &\cdot \left\{ -4\pi \int_0^\infty dr r^2 j_0(pr) U(r, k^2) \right\} \end{aligned} \quad (32)$$

For this spinless case we may write $C^B(\vec{k}', \vec{k}) \equiv C^B(p, k)$. The first term at the right hand side can now be identified with $\frac{M}{E_k}$, thereby defining $N(|\vec{r}' - \vec{r}|)$. Expressing the Feynman amplitude in terms of the scattering amplitude (Appendix chapter III, page 148), Wong's result :

$$M^B(p, k) = -M \int_0^\infty dr r^2 j_0(pr) U(r, k^2) \quad (33)$$

is obtained. By inversion we can express the potential $U(r, k^2)$ as the Fourier-Bessel transform of the scattering amplitude. In the general case, including spin, we can express each of the five matrix elements $^{*)}M^B(p, k)_{S m_S \rightarrow S m'_S}$ as a sum of the Fourier-Bessel transforms of the five potential terms in the general Okubo-Marshak form (chapter I, page 34). These expressions can also be inverted to obtain the five potentials as a sum of Fourier-Bessel transforms of $M^B(p, k)_{S m_S \rightarrow S m'_S}$. The matrix elements can be calculated using simple expressions of these in terms of the helicity amplitudes, which are given for instance in reference 48.

An important consequence of the inclusion of the relativistic phase-space element is, that it leads to a non-vanishing expression for the quadratic spin-orbit term.

Erkelenz et. al. (ref. 85) have used the OBE-potential obtained in this manner to calculate phase parameters. For simplicity they left out the \vec{k} -dependence of the potential terms, except for the central potential.

In their model the mesons π , η , σ , ω , δ and an iso-scalar scalar are considered. To regularize the potential singularities at $r=0$, they introduced a zero cut-off radius d , within which the potential is put to zero.

*) Wong uses the spin triplet elements $M_{0 \rightarrow 0}$; $M_{1 \rightarrow -1}$; $M_{1 \rightarrow 1}$; $M_{0 \rightarrow 1}$; $M_{1 \rightarrow 0}$; and the spin singlet element

The masses of the π , η , ρ and ω were pre-determined. The free parameters in their model are the coupling constants, two cut-off radii (one for L even and one for L odd), and the masses for the δ -meson (iso-vector scalar) and the iso-scalar scalar meson. The parameters were adjusted to fit the phase parameters for $L=0$, $L=1$ and $L=2$.

In their model 11 parameters are used. The fit to the same set of 36 data, as employed by Ueda and Green, yields the reduced $\chi^2=1.6$. This is the same as obtained in the Ueda and Green model. An explicit study of the quadratic spin-orbit term was made by doing the fit without this term. The agreement of the critical 1D_2 -phase-shift to the data is indeed slightly worsened (see figure 25, page 122). They found a considerable worsening of the fit to the coupling parameter ϵ_1 between the 3S_1 - and 3D_1 -states. In figure 23 below, the coupling parameters are shown, and table 11, page 146 gives the values for masses and coupling constants.

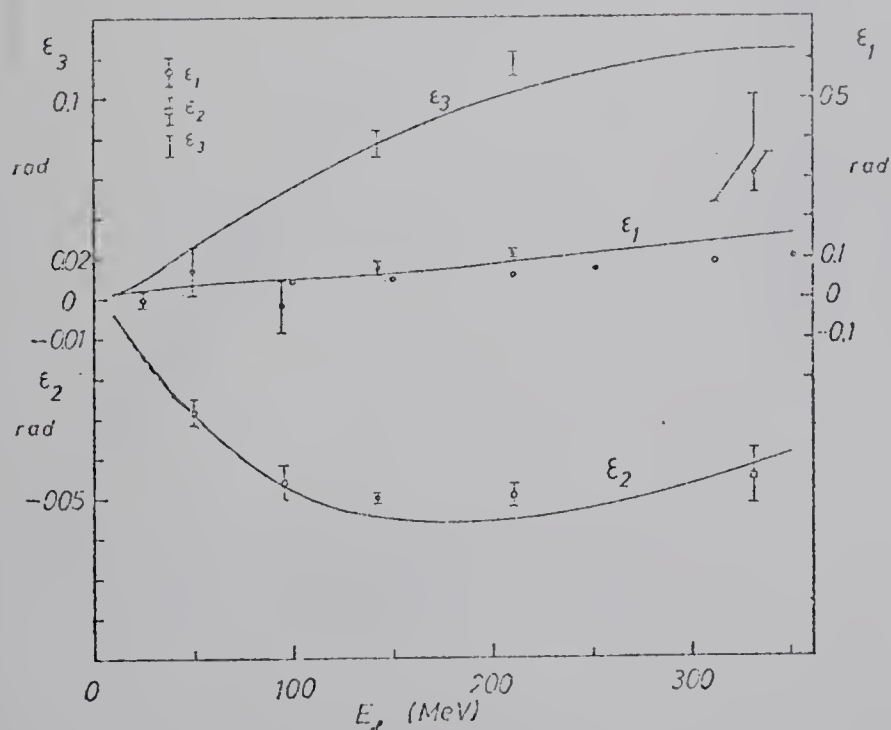


figure 23 :

the coupling parameters in the Erkelenz et. al. model; experimental points are taken from a Livermore analysis (ref. 98)

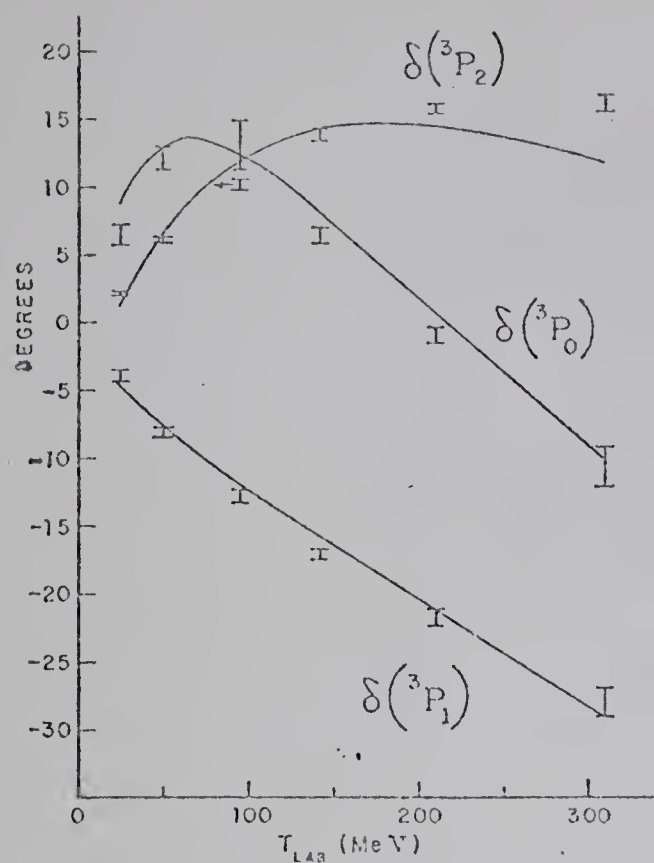


figure 24 :

the triplet P-phase-shifts of the Bryan-Scott model; empirical values are taken from a Livermore analysis (ref. 87)

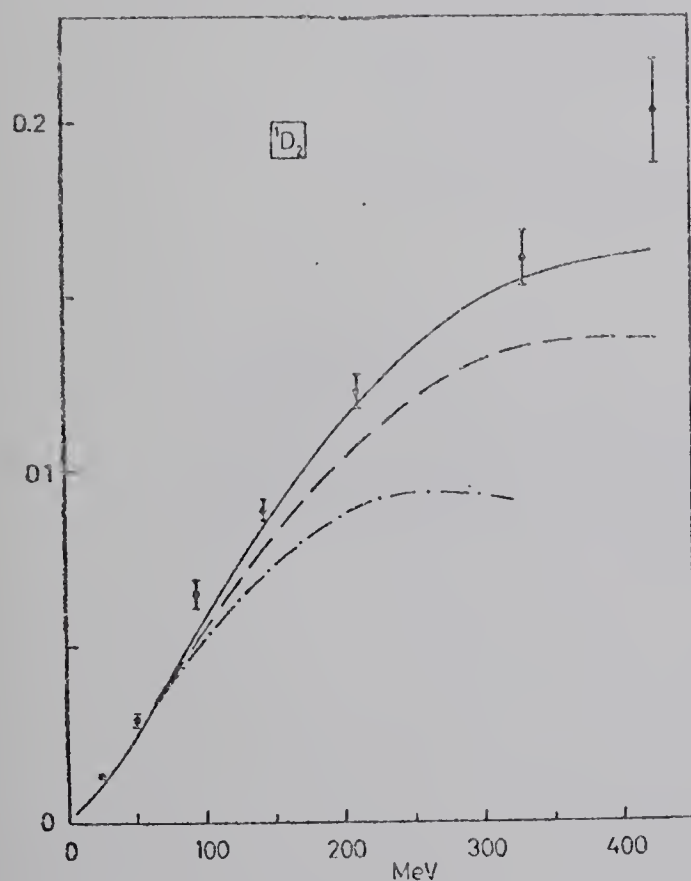


figure 25 :

comparison of the 1D_2 -phase-shift, calculated from the Erkelenz model (solid curve); the Erkelenz model omitting the L_{12} potential (dashed curve) and the Bryan-Scott model (dash-dot line); empirical values are taken from a Livermore analysis (ref. 78)

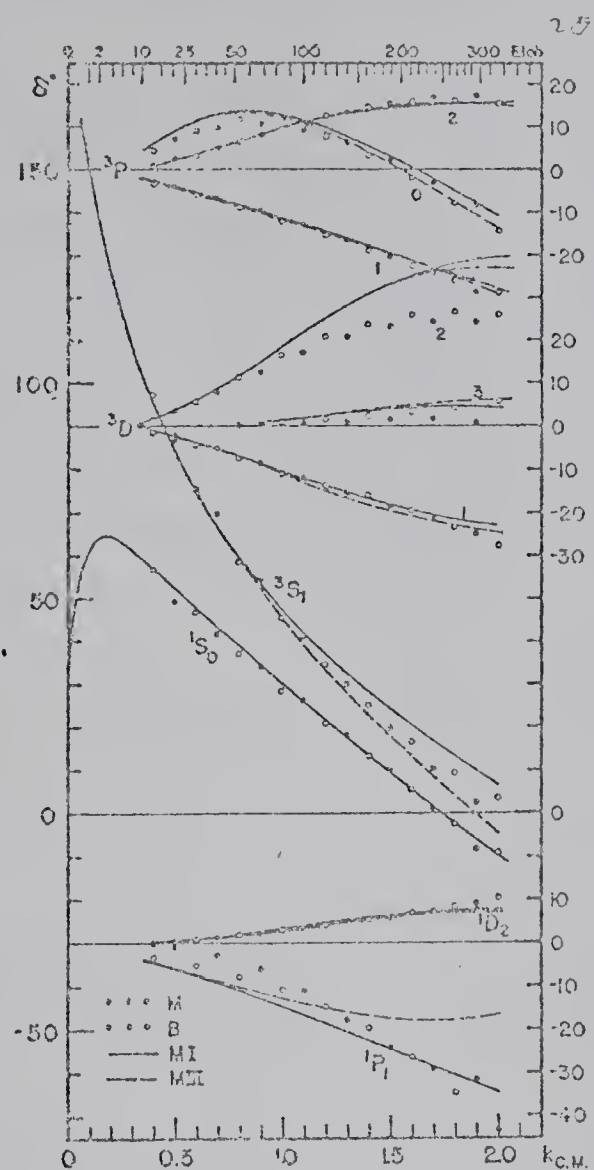


figure 26 :

phase-shifts from the Ueda-Green model (solid curve); open circles represent phase-shifts of Livermore (ref. 93) and Yale (ref. 94) respectively

4) Momentum Space OBE-Potentials

A coordinate space representation of an OBE-potential suffers from two defects. The discussion starts with a momentum space representation, and a Fourier transform is applied to find the coordinate space representation. In order to do so we had to take the non-relativistic limit, and secondly we had to confine ourselves to the on-shell matrix elements of the potential, thereby losing the off-shell information contained in the Feynman amplitude. This off-shell behaviour is of interest also for elastic scattering if we use the Lippmann-Schwinger equation. Typical relativistic effects as the meson retardation are not included due to the on-shell assumption $p_0=0$ (see discussion following equation 13). These approximations can be circumvented if we perform the calculations in momentum space. Most recent OBE-models (after 1970) are indeed formulated in momentum space. Nevertheless as long as the OBE-potential is used in conjunction with a Schrödinger equation, we introduce a non-relativistic element in the discussion. The Feynman amplitude in the OBE-model is a solution of the ladder approximated Bethe-Salpeter equation. This equation is very complicated to solve due to its four dimensional character, therefore one relies on approximations.

An approximate relativistic description can be obtained by using Wong's relativistic phase-space element in the Lippmann-Schwinger equation :

$$C(\vec{k}', \vec{k}) = U(\vec{k}', \vec{k}) + n_1 \int d^3q \frac{M}{E_q} U(\vec{k}', \vec{q}) C(\vec{q}, \vec{k}) (q^2 - k^2 - i\epsilon)^{-1} \quad (34)$$

In the OBE-model the potential $U(\vec{k}', \vec{k})$ is related to the Born term of the Feynman amplitude by equation 11.

n_1 is a constant that will be determined below. The solution of this equation should be a good approximation of the Feynman amplitude $C(\vec{k}', \vec{k})$, which is a solution of the ladder approximated Bethe-Salpeter equation. A first requirement that we should impose is that the solution $C(\vec{k}', \vec{k})$ of equation (34) satisfies the unitarity condition for the Feynman amplitude. This principle is known in the literature as 'minimal requirement of relativity' (ref. 95). The unitarity condition in terms of $C(\vec{k}', \vec{k})$ is found from :

$$T(\vec{k}', \vec{k}) = \sum_L (2L+1) (e^{2i\delta_L(k)} - 1) P_L(\cos\theta) , \quad (35)$$

the relation (1), and :

$$C(\vec{k}', \vec{k}) = \frac{4\pi E_k}{M^2} M(\vec{k}', \vec{k}) \quad (36)$$

A derivation of (36) is given in the appendix attached to this chapter. One obtains :

$$C(\vec{k}', \vec{k}) = \left(\frac{4\pi}{M}\right)^2 \frac{E_k}{k} \sum_L (2L+1) e^{i\delta_L(k)} \sin\delta_L(k) P_L(\cos\theta) \quad (37)$$

Therefore :

$$\text{Im}C(\vec{k}', \vec{k}) = \frac{M^2}{(4\pi)^3} \frac{k}{E_k} \int d\Omega_q |C(\vec{k}, \vec{q})|^2 \quad (38)$$

Let us check if the solution $C(\vec{k}', \vec{k})$ of (34) satisfies this relation. One can derive from (34) the formula :

$$C(\vec{k}, \vec{k}) - C^*(\vec{k}, \vec{k}) = \int d^3q C^*(\vec{k}, \vec{q}) \{g(\vec{k}, \vec{q}) - g^*(\vec{k}, \vec{q})\} C(\vec{k}, \vec{q}) \quad (39)$$

where :

$$g(\vec{q}, \vec{k}) = -n_1 \frac{M}{E_q} (q^2 - k^2 - i\epsilon)^{-1} \quad (40)$$

where $q \equiv |\vec{q}|$
 $k \equiv |\vec{k}|$

Details of this derivation can be found for instance in reference 100. An easy way to verify (39) is to use the symbolic relation $C = V + VgC$ and to eliminate V from this and the similar relation for C^* . From (39) and (40) follows :

$$\text{Im}C(\vec{k}, \vec{k}) = \int d^3q C^*(\vec{k}, \vec{q}) \left\{ n_1 \frac{M}{E_q} \Pi \delta(q^2 - k^2) \right\} C(\vec{k}, \vec{q}) \quad (41)$$

Using $d^3q = q E_q dE_q d\Omega_q$ and $\delta(q^2 - k^2) = \frac{1}{2E_q} \delta(E_q - E_k)$ we obtain:

$$\text{Im}C(\vec{k}, \vec{k}) = n_1 \frac{\Pi}{2} \frac{Mk}{E_k} \int d\Omega_q |C(k, q)|^2 \quad (42)$$

Comparing (38) and (42) we notice that agreement with the unitarity condition is established for $n_1 = \frac{\Pi}{2} \frac{1}{(2\Pi)^4}$.

One may write (34) in the usual Lippmann-Schwinger form by defining :

$$V(\vec{k}', \vec{k}) = \frac{M}{E_k} U(\vec{k}', \vec{k}) \quad \text{and} \quad M'(\vec{k}', \vec{k}) = \frac{4\Pi}{M} M(\vec{k}', \vec{k}) \quad (43)$$

Substitution of (43) into (34) gives :

$$M'(\vec{k}', k) = V(\vec{k}', \vec{k}) + n_1 \int d^3q V(\vec{k}', \vec{q}) M'(\vec{q}, \vec{k}) (q^2 - k^2 - i\epsilon)^{-1} \quad (44)$$

Clearly the configuration space representation of $V(\vec{k}', \vec{k})$ is just the non-local potential (31), introduced by Wong.

Next we briefly consider the relation of the modified Lippmann-Schwinger equation (34) with the Bethe-Salpeter equation. Using the notation of Partovi and Lomon (ref. 96), the Bethe-Salpeter equation is :

$$C(k'|W) = K(k'|W) + \int d^4q K(k'q|W)G(q|W)C(qk|W) \quad (45)$$

where k' and k are half of the relative 4-momenta, respectively after and before the interaction. W is half of the total 4-momentum. $K(k'|W)$ is the kernel which in our discussion just represents the OBE-diagram. $G(q|W)$ is the propagator for the two (free) nucleons. For the spinless case this is :

$$G(q|W) = \frac{i}{(2\pi)^4} \{ (W+q)^2 + M^2 + i\epsilon \}^{-1} \{ (W-q)^2 + M^2 + i\epsilon \}^{-1} \quad (46)$$

Various ways to approximate the Bethe-Salpeter equation by a three dimensional expression have been proposed by a number of authors (ref. 4). We shall briefly consider the approach of Blanckenbecler and Sugar (ref. 113). The intermediate state nucleons can be in negative energy states. The assumption made by Blanckenbecler and Sugar is that for elastic low energy scattering the contribution of the intermediate negative energy states to the final states are negligible. These negative energy states appear in the formalism through the singularities of $G(q|W)$ along the negative real axis in the energy plane. Blanckenbecler and Sugar proposed therefore a description in which $G(q|W)$ is replaced by a propagator $g(q|W)$, which is singular along the positive real axis only. The particular choice for $g(q|W)$ that they made is :

$$g(q|W) = n_2 \int_0^\infty dq'^2 (q'^2 - k^2 - i\epsilon)^{-1} \delta^+ \{ (W'+q)^2 + M^2 \} \delta^+ \{ (W'-q)^2 + M^2 \} \quad (47)$$

δ^+ signifies that only the positive root of the argument should be included. Furthermore $W'^2 = q'^2 + M^2$. n_2 is a constant that will be determined later. The integration in (45) can be performed using :

$$\delta^+ \{ (W' + q)^2 + M^2 \} \delta^+ \{ (W' - q)^2 + M^2 \} = \frac{1}{4E_q} \delta(q_0) \delta(q'^2 - q^2) \quad (48)$$

The result is :

$$g(q|W) = \frac{n}{4E_q} (q^2 - k^2 - i\epsilon)^{-1} \quad \text{and} \quad k_0 = k'_0 = q_0 = 0 \quad (49)$$

Comparing (47) and (38) we notice that the two are equivalent if we put $n = -4Mn_1$. The generalization of this result to spin- $\frac{1}{2}$ particles is straight forward and consists essentially in multiplying the integrand in (47) with a positive energy projection operator (ref. 96).

Holinde et. al. (ref. 97) considered in their OBE-model a modified form of the potential (43) :

$$V(\vec{k}', \vec{k}) = \left(\frac{M}{E_{k'}} \right)^{1/2} U(\vec{k}', \vec{k}) \left(\frac{M}{E_k} \right)^{1/2} \quad (50)$$

Consequently in their description $M'(\vec{k}', \vec{k})$ is related to $C(\vec{k}', \vec{k})$ by :

$$M'(\vec{k}', \vec{k}) = \left(\frac{M}{E_{k'}} \right)^{1/2} C(\vec{k}', \vec{k}) \left(\frac{M}{E_k} \right)^{1/2} \quad (51)$$

Furthermore they do not relate their OBE-potential to C_B as in equation (11), but use : (52)

$$\langle s'_1, s'_2 | U^i(\vec{k}', \vec{k}) | s_1, s_2 \rangle = \bar{u}_{s'_1}(-\vec{k}') \Gamma_i u_{s_1}(-\vec{k}) P_i \bar{u}_{s'_2}(\vec{k}') \Gamma_i u_{s_2}(\vec{k})$$

The interesting feature of the modified form (50) is that it allows for the possibility :

$$E_{k'} \neq E_k \quad . \quad (53)$$

In this way the effects of the meson retardation are included. We notice that in the Blanckenbecler-Sugar form the energies of both nucleons are equal in the intermediate - and final states.

A convenient method to calculate the phase parameters from equation (33) is to consider only the principal value of the integral, in which case the solution is $\frac{(4\pi)^2}{2Mk}K(\vec{k}',\vec{k})$, where the K-matrix is related to phase parameters as shown in equations (4) and (5). Details of the numerical solution of the integral equation in terms of the K-matrix are given in reference 95.

The Holinde et. al.-model involves the exchange of the π , ρ , ω , ϕ and δ -mesons, and in addition an iso-scalar scalar. All meson masses were predetermined, except the one of the scalar meson. Furthermore the ratio of the coupling constants g_ω and g_ϕ for the ω and the ϕ -mesons was fixed at 2:1. (Erroneously this ratio is considered to be justified from SU(3) (ref. 111).)

Although the requirement of minimal relativity improves the convergence of the potential $V(\vec{k}',\vec{k})$, the integral equation is not of Fredholm type and a cut-off procedure has to be used. For simplicity, a form factor of the type (27) is used for all mesons. For the vector mesons, two form factors were used, one for each vertex. The values were chosen the same for each meson. The 9 coupling constants were all adjusted, so in total 12 free parameters were used. Values for the masses and coupling constants are shown in table 12, page 147. The free parameters were adjusted by fitting the deuteron binding energy and the phase parameters of a Livermore analysis (ref. 98). The reduced goodness of fit for 89 empirical phase parameters is 2.5. In their coordinate space OBE-model (ref. 85), the value 2.7 was obtained for this same set of data. Unfortunately one can not directly conclude from this that momentum space potentials are superior, as in the momentum space model a ϕ -meson was included and not in the configuration space model. However for the S-wave phase-shifts significantly better results were obtained with the momentum space model. These were

predicted too high in the configuration space model. The required lowering of the phase-shifts is due to taking the off-shell behaviour of the potential into account. This was shown by repeating the calculation in which $E_{k'}$ was set equal to E_k . The results are shown in figure 27. The effect of the meson retardation was studied separately by setting $E_{k'} = E_k$ in the meson propagators only, again the phase-shifts are lowered, implying an additional attraction. This is shown in figure 28. Differences of about 30% were obtained. This shows that the meson retardation is an important effect. Nevertheless it has been shown that a comparable fit is possible leaving meson retardation out of account, by redefining the free parameters (ref. 3). Finally we mention that the 1D_2 and the 3D_2 could not be fitted satisfactory.

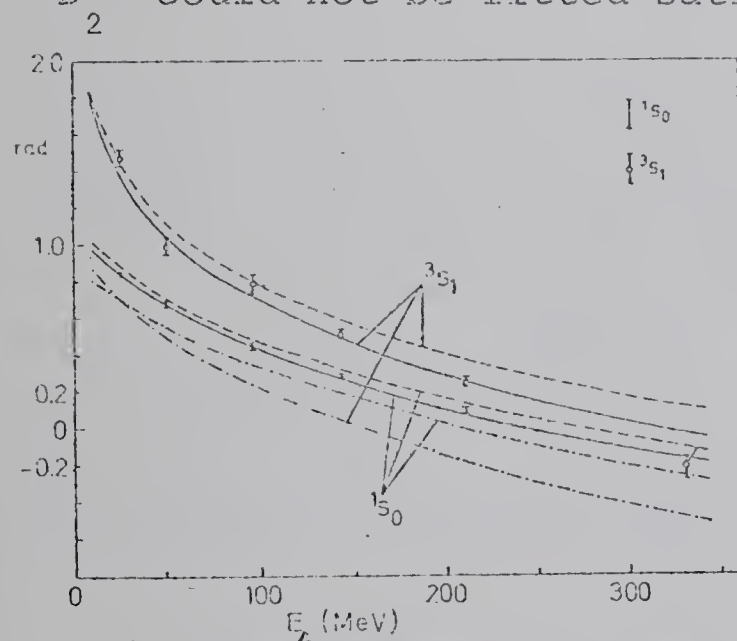


figure 27 :

the solid curve denotes the q-space results; the dashed curve represents the r-space OBE (ref.85), the dash-dot curve the on-shell approximation of the q-space OBE; empirical points are taken from reference 98

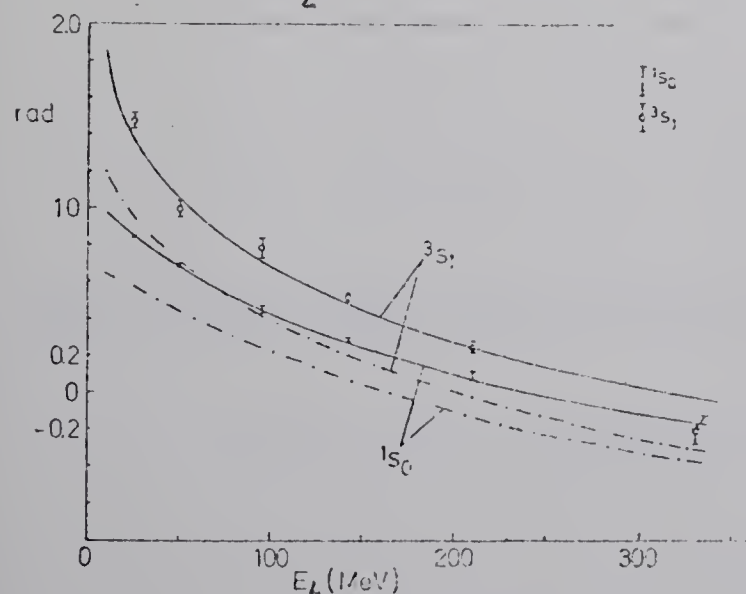


figure 28 :

S-wave phase-shifts with and without meson representation, represented by the solid - and dash-dot curves respectively

5) Unitarization by Dispersion Relations

The unitarity condition can be incorporated in the OBE-theory by making use of dispersion relations.

The unitarity condition has a simple form in a partial wave analysis. In chapter II we discussed dispersion relations for the causal amplitude A . Dispersion relations for the partial wave causal amplitudes can be obtained from these. The dispersion relations together with the unitarity condition lead to non-linear integral equations for the partial wave amplitudes. These non-linear equations can be converted to a linear form by applying the N/D -method, introduced by Chew and Mandelstam (ref. 99). The partial wave dispersion relations and the N/D -method are the subjects of this section. To avoid the technical complications due to the nucleon spin and isospin, we study the case of spinless - and neutral 'nucleons'. We indicate how the extension to the realistic case can be made. As an example of the dispersion relation approach we briefly review the Scotti-Wong model (ref.104).

In the discussion of partial waves it is convenient to use, instead of the Mandelstam variables, the cosine of the scattering angle x and the square of the momentum in the cm-system v , to describe the kinematics. The relation between x and v and the Mandelstam variables is :

$$\begin{aligned} s &= 4(M^2 + v) \\ t &= -2v(1-x) \\ u &= -2v(1+x) \end{aligned} \tag{54}$$

The causal amplitude can be written as :

$$A(v, x) = n_3 \sum_L \left(L + \frac{1}{2}\right) A_L(v) P_L(x) \tag{55}$$

where

$$A_L(v) = \left(\frac{M^2+v}{v}\right)^{1/2} e^{i\delta_L(v)} \sin\delta_L(v) \quad (56)$$

This result is given in the review article by Furuichi (ref. 52) and is obtained by using the partial wave expansion of the amplitude T (34), the relation between T and M (2), the relation between M and C (appendix chapter III (4)) and the relation between C and A (chapter II(75)). The factor n_3 turns out to be $n_3 = (4\pi/M)^2$. We assume that the amplitude $A(v,x)=A(s,t;u)$ satisfies the fixed t dispersion relation (chapter II(118)). The dispersion relation for $A_L(v)$ can be found from this dispersion relation by applying :

$$A_L(v) = (4\pi/M)^2 \int dx P_L(x) A(v,x) \quad (57)$$

The integration over x introduces cuts in the complex v -plane. The cut structure is found by inspection of the denominators in the dispersion relation of $A(s,t;u)$. The denominators of the pole terms have the form :

$$u-m^2 = -2v(1+x)-m^2 \quad (58)$$

This quantity can become zero for some value of x if :

$$v < -v_0, \quad v_0 \equiv \frac{m^2}{4} \quad (59)$$

The integration of the pole term over x generates a left hand cut in the v -plane, specified by (59).

Similarly the denominators $s'-s$ and $u'-u$ in the dispersion relation for $A(s,t;u)$ give rise to the cuts :

$$s'-s \text{ and } 4M^2 < s' \rightarrow \text{right hand cut } 0 < v \quad (60)$$

$$u'-u \text{ and } 4m^2 < u' \rightarrow \text{left hand cut } v < -m^2 \quad (61)$$

This cut structure is illustrated in figure 29.

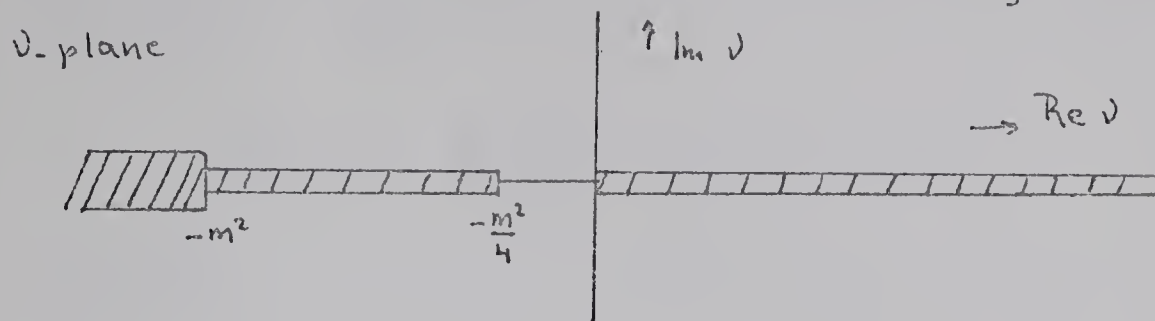


figure 29 : the cut structure of the v -plane

We notice that the second left hand cut for $v < -m^2$ is due to the exchange of two or more mesons. In the OBE-model we leave this out of account.

We assume for simplicity that $A_L(v)$ vanishes rapidly for large values of v . From the cut structure of the v -plane and the Cauchy theorem, we can write down the dispersion relations for $A_L(v)$:

$$A_L(v) = \frac{1}{\pi} \int_{-\infty}^{-v} dv' (v' - v - i\epsilon)^{-1} \text{Im} A_L(v') + \frac{1}{\pi} \int_0^{\infty} dv' (v' - v - i\epsilon)^{-1} \text{Im} A_L(v') \quad (62)$$

The first integral represents the OBE-contribution. The second one is due to the s -channel unitarity. If we assume elastic unitarity, then from (56) we find :

$$v > 0 : \quad \text{Im} A_L(v) = \left\{ \frac{v}{v + M^2} \right\}^{1/2} |A_L(v)|^2 \quad (63)$$

$\text{Im} A_L(v)$ for negative values of v can be calculated explicitly in the OBE-model. For a pole term in $A(v, x)$ we obtain :

$$A_L(v) = - \left(\frac{4\pi}{M} \right)^2 \int_{-1}^1 dx g^2 \{ 2v(1+x) + m^2 \}^{-1} P_L(x) \quad (64)$$

The pole for $x = -1 - \frac{m^2}{2v}$ gives rise to the imaginary part of $A_L(v)$. From :

$$\text{Im}A_L(v) = \frac{1}{2i} \{A_L(v+i\varepsilon) - A_L(v-i\varepsilon)\} \quad (65)$$

we find :

$$v < 0 : \quad \text{Im}A_L(v) = -\frac{\pi}{2} \left(\frac{4\pi g}{M}\right)^2 \frac{(-1)^L}{v} P_L\left(1 + \frac{m^2}{2v}\right) \quad (66)$$

If we substitute the results (63) and (66) into the dispersion relation, we obtain an inhomogeneous non-linear integral equation for $A_L(v)$:

$$A_L(v) = g_L(v) + \frac{1}{\pi} \int_0^\infty dv' \left(\frac{v'}{v'+M^2}\right)^{1/2} |A_L(v')|^2 (v'-v-i\varepsilon)^{-1} \quad (67)$$

In this equation we have denoted the pole contribution by the function $g_L(v)$. To linearize and subsequently solve this equation we can use the N/D-method. In the N/D-method we write $A_L(v)$ as :

$$A_L(v) = N_L(v)/D_L(v) \quad (68)$$

where $N_L(v)$ is analytic in the v -plane except for the branch cut from $-\infty$ to $-v_0$. $D_L(v)$ is analytic in the v -plane except for the branch cut from 0 to ∞ . Both these cuts are taken along the real axis. From the observation that in the elastic region :

$$\text{Im}A_L^{-1}(v) = -\{v(v+m^2)^{-1}\}^{1/2} \quad (69)$$

we can deduce the following form of the unitarity requirement :

$$\text{Im}D_L(v) = -\{v(v+m^2)^{-1}\}^{1/2} N_L(v)\theta(v) \quad (70)$$

Using (67) one sees :

$$\text{Im} N_L(v) = \text{Im } g_L(v) D_L(v) \theta(-v-v_0) \quad (71)$$

We assume appropriate asymptotic behaviour so that both $N_L(v)$ and $D_L(v)$ satisfy once-subtracted dispersion relation. We use the results (70) and (71), and the normalization $D_L(0)=1$. The subtraction point is taken to be $v=0$. The resulting expressions are :

$$D_L(v) = 1 - \frac{v}{\pi} \int_0^\infty \frac{dv'}{v'} \left(\frac{v'}{v'+M^2} \right)^{1/2} N_L(v') (v'-v-i\epsilon)^{-1} \quad (72)$$

$$N_L(v) = A_L(0) + \frac{v}{\pi} \int_{-\infty}^0 \frac{dv'}{v'} \text{Im } g_L(v') D_L(v') (v'-v-i\epsilon)^{-1} \quad (73)$$

One notices that the subtraction constant $A_L(0)$ can be interpreted as the scattering length for $L=0$. The solution of these coupled equations proceeds as follows : First we combine the two equations by eliminating $N_L(v)$. This results in an equation for $D_L(v)$, which can be solved for negative values of v . It is not difficult to see that this expression can be written as :

$$v < 0 : \quad D_L(v) = 1 - A_L(0) \gamma(v) - \frac{v}{\pi} \int_{-\infty}^0 dv' \frac{1}{v'} \text{Im } g_L(v') \cdot D_L(v') \frac{\gamma(v') - \gamma(v)}{v' - v} \quad (74)$$

where :

$$\gamma(v) = \frac{1}{\pi} \int_0^\infty dv' \{v' (v'+M^2)\}^{-1/2} (v'-v)^{-1} \quad (75)$$

The important observation that one can make is that the integration (75) is convergent. The explicit form of $\gamma(v)$ is :

$$\gamma(v) = \left(\frac{v}{v+M^2}\right)^{1/2} \ln \frac{\{v/(v+M^2)\}^{1/2} + 1}{\{v/(v+M^2)\}^{1/2} - 1} \quad (76)$$

The convergence of this integral is the reason to consider a once-subtracted dispersion relation for $D_L(v)$. Furthermore we notice the importance of the factor $\{v/(v+M^2)\}^{1/2}$ in this respect. For $v' < -v_0$ the integrand in (74) is non-singular, therefore we can solve $D_L(v)$ for $v < 0$, for instance by iteration. (Another possibility is matrix inversion. In that way the solution for $\text{Re}D_L(v)$ can be obtained for $v > 0$ also). Having solved $D_L(v)$ we can calculate $N_L(v)$ for every value of v from equation (73). Finally $A_L(v)$ is obtained.

The phase-shifts are found as follows :

The unitarity condition (56) can be written as :

$$v > 0 : \quad \left(\frac{v}{v+M^2}\right)^{1/2} A_L(v) = \frac{K_L}{2 - iK_L} \quad (77)$$

where $K_L = 2 \tan \delta_L(v)$

Using (70) one can check that K_L is indeed a real quantity. The explicit expression is :

$$K_L = 2 \left(\frac{v}{v+M^2}\right)^{1/2} \frac{N_L(v)}{\text{Re}D_L(v)} \quad (78)$$

where :

$$\text{Re}D_L(v) = 1 - v P \int_0^{\infty} dv' \{v'(v'+M^2)\}^{-1/2} N_L(v') (v'-v)^{-1} \quad (79)$$

In the foregoing discussion we illustrated how the phase-shifts can be found by applying dispersion relations. However the procedure that we outlined is not quite satisfactory. Problems arise with the threshold behaviour in the OBE-model. First we show that the exact partial wave amplitude in the dispersion relation formalism does have the

correct threshold behaviour. Having done this, we show that this behaviour is not satisfied in the OBE-model.

According to a theorem in potential scattering, the partial wave amplitude has the following threshold behaviour (ref.101) :

$$\lim_{v \rightarrow 0} A_L(v) \sim v^L \quad (80)$$

This result can be reproduced by the dispersion relations. To see this, we consider the fixed- s dispersion relation for the causal amplitude. After substitution of the variables v and x for t and u , we find the following form for the partial wave amplitudes :

$$\begin{aligned} A_L(v) = & \left(\frac{4\pi}{M}\right)^2 \int_{t_0}^{\infty} dt' \operatorname{Im} A_t(t', v) \int_{-1}^1 dx P_L(x) \{t' + 2v(1-x) - i\varepsilon\}^{-1} + \\ & + \left(\frac{4\pi}{M}\right)^2 \int_{u_0}^{\infty} du' \operatorname{Im} A_u(u', v) \int_{-1}^1 dx P_L(x) \{u' + 2v(1+x) - i\varepsilon\}^{-1} \end{aligned} \quad (81)$$

For the moment we leave the pole terms out of the discussion. If v is small enough, we can expand the denominators in powers of $v(1 \pm x)$. The factors $(1 \pm x)^n$ can be expanded as linear combinations of the Legendre polynomials $P_m(x)$ for $m \in \{0, 1, \dots, n\}$. Using the orthonormality of the Legendre polynomials one obtains the result that the lowest order contribution in the power series in v is v^L . The v -dependent integrals over u' and t' are constants for $v \rightarrow 0$, therefore the expected threshold behaviour (80) is obtained. We notice that it is vital in this argument that u_0 and t_0 are finite. We consider now the dispersion relation (62). The foregoing arguments clearly do not apply for the s -channel unitarity integral. Indeed it does not have the correct threshold behaviour

(ref. 52). Therefore we conclude that the left hand side contribution does not have the limit (80), because the sum of terms in (64) does have the correct threshold behaviour (in the exact case). Cancellations between terms occur to the effect that the limit (82) is obtained.

In the OBE-model we only consider the pole contributions to the first integral in (62). The explicit form is :

$$\begin{aligned} v > 0 \quad ; \quad g_L(v) &= - \frac{1}{2} \left(\frac{4\pi g}{M} \right)^2 (-1)^L \int_{-\infty}^{-m^2/4} dv' \{v'(v'-v)\}^{-1} P_L \left(1 + \frac{m^2}{2v'} \right) \\ &= + \left(\frac{4\pi g}{M} \right)^2 (-1)^L \frac{1}{v} Q_L \left(1 + \frac{m^2}{2v} \right) \end{aligned} \quad (82)$$

where :

$$Q_L \left(1 + \frac{m^2}{2v} \right)$$

is the Legendre function of the second kind.

From the asymptotic behaviour of $Q_L \left(1 + \frac{m^2}{2v} \right)$ follows :

$$\lim_{v \rightarrow 0} g_L(v) \sim v^L \quad (83)$$

Consequently $g_L(v)$ can not provide the cancellations necessary to obtain the correct threshold behaviour for A_L in the OBE-model.

A simple solution for this problem, suggested by Kantor (ref. 102) is to use L-subtracted dispersion relations :

$$A_L(v) = g_L(v) + \frac{1}{\pi} v^L \int_0^\infty dv' \frac{1}{v'^L} \left(\frac{v'}{v'+M} \right)^{1/2} |A_L(v')|^2 (v'-v-i\epsilon)^{-1} \quad (84)$$

where we have put :

$$A_L(0) = \{v^{-1} A_L(v)\}_{v=0} = \dots = \{v^{-L+1} A_L(v)\}_{v=0} = 0 \quad (85)$$

Although the form (84) has the correct threshold behaviour we notice that $A_L(v)$ behaves asymptotically as :

$$A_L(v) \sim v^{L-1} \quad (86)$$

which is in violent conflict with the Froissart-Gribov bound for the asymptotic behaviour of the total cross section (ref. 60). Another difficulty is that this subtraction causes a divergent solution, when we solve (84) by iteration (ref. 52).

Scotti and Wong (ref. 104) avoid these problems by multiplying the integrand in (84) for $L \geq 1$ by :

$$\{(v' - v_c)(v - v_c)^{-1}\}^{L-1} \quad (87)$$

where v_c is a negative real constant. In doing so they obtain both the correct threshold behaviour and the correct asymptotic behaviour.

As we discussed in chapter I, section 3, the effect of the interaction is larger for the lower partial waves. We expect therefore that the importance of the unitarity integral in (84) is more significant in the lower than in the higher partial waves. Scotti and Wong proposed the following treatment :

- for $L > 6$: the unitarity integral is neglected
- for $L = 3, 4, 5$: $A_L(v')$ in the integral is replaced by the one pion exchange contribution $g_L^\pi(v)$
- for $L = 1, 2$: the full equation (84) modified as discussed above, is solved by using the N/D-method
- for $L = 0$: one subtraction at $v = 0$ is introduced and the equation is also solved using the N/D-method

In the realistic case we should include the spin and iso-spin of the nucleons and the various exchanged mesons. To extend the discussion to that case, we have to use ten amplitudes, five for the various transitions between the spin states and one for each of the two iso-spin states. Furthermore we notice that couplings between the various L -states occur. The N/D -method, as we discussed it, should therefore be generalized to handle the coupled channel problem. Bjorken and Nauenberg (ref. 105) have shown that such a generalization can be made. To this end one writes the scattering matrix as a product of a matrix N and the inverse of a matrix D . Dispersion relations can be written for the matrix elements of these matrices. Details of the method can be found in the original paper of Bjorken and Nauenberg or in the Scotti and Wong article.

In their model Scotti and Wong take into consideration the π , η , ρ , ω and ϕ -meson and furthermore an iso-scalar scalar meson. The subtraction constant for the S -waves is adjusted to the empirical scattering lengths :

$$\begin{aligned} a_{np}^{I=0} &= 5.4 \text{ fm} \\ a_{np}^{I=1} &= -23.74 \text{ fm} \\ a_{pp} &= -7.7 \text{ fm} \end{aligned} \tag{88}$$

The meson masses, except those of the large-width ρ -meson and the fictitious iso-scalar scalar, are predetermined, as is the pion-nucleon coupling constant. The divergence of the amplitudes for vector mesons was handled by adding an exponentially decreasing factor, such that at high energies the Regge behaviour is obtained.

In this procedure they introduced three adjustable parameters, one for each of the three vector mesons in their model. The pole v_c was taken to be the same for all partial waves. In total the model involves 12 free parameters. Table 13 gives masses and coupling constants (page 147).

Another peculiarity of their method is that they did not try to fit calculated phase parameters in their model with a phase-shift analysis. Instead they calculated observables directly and fitted those with the empirical data. The fit was quite good. A calculation of the phase parameters yielded a reduced χ^2 of 2.5 for 377 p-p data points (ref. 106).

A disadvantage of their method is the inclusion of the $(L-1)$ th order pole at v_c , the physical origin of which is rather obscure. Furthermore the Regge type cut-off procedure caused difficult computer calculations (ref. 20). It was pointed out by Signell that a Coulomb interference correction which was used, over estimated the effect (ref. 20).

One may notice that the introduction of the unphysical pole at v_c is not necessary for the S-waves. It is particularly in this case that the partial wave dispersion relations have been applied successfully by Noyes and Wong (ref. 107) and others. These analyses suggest that by adjusting the scattering lengths using a once-subtracted dispersion relation, we can reproduce the deuteron binding energy and the S/D-mixing ratio (ref. 52).

6) Concluding Remarks

The various OBE-models have in common that, for the outer range, the interaction is described by the pion exchange and for the short range repulsion the ω -meson is indispensable. Furthermore the intermediate range attraction can not be reproduced accurately without an iso-scalar scalar of about 500 MeV. Erkelenz (ref. 3) points out that to fit the S-phase-shifts, the δ -meson should be used. The characteristic bending down of the 3P_0 phase-shift requires the inclusion of the ρ -meson. Minor improvements are possible by including other mesons such as the η and the Φ .

All OBE-models take the meson-nucleon coupling constants as parameters to be searched over. From other sources the πN coupling constant is well established : $g_{\pi}^2 = 14.50 \pm 0.50$. Values of the ρN coupling constants are reported in the range $g_{\rho}^2 = 0.5 \pm 0.2$ and $f_{\rho}/g_{\rho} = 3.80 \pm 0.50$ (ref. 4). Quark model considerations lead to $g_{\rho}^2 : g_{\omega}^2 = 1:9$ and $g_{\Phi}^2 = 0$ (ref. 4; see also ref. 111). So $g_{\omega}^2 = 5 \pm 2$. From the very small iso-scalar anomalous magnetic moment of the nucleon one knows that $f_{\omega}/g_{\omega} \approx 0$. SU(3) predicts for the pseudo scalars $g_{\pi}^2 : g_{\eta}^2 = 25:3$, so $g_{\eta}^2 \approx 2$. Virtually nothing is known about the coupling constants of the scalar mesons (ref. 4; see however ref. 112). From the tables 8 - 13 on page 145 - 147, one notices that general agreement exists about the πN coupling constant. Also the ratio $g_{\rho}^2 : g_{\omega}^2 = 1:9$ is reproduced with acceptable accuracy, except by Scotti and Wong. Especially the results of Ueda and Green agree with the values given above. In these comparisons, one should keep in mind that the OBE-values will depend on the meson masses that are chosen, which differ in the various models and also on the cut-off procedure that is employed. The unphysical values in the Scotti-Wong model are probably due to the 4 purely phenomenological parameters that are used. Erkelenz et. al. (ref. 108) pointed out that if one wants

to compare the values of the coupling constants, the residues of the pole terms, with empirical values one has to renormalize the coupling constants. For a form factor of the type (27), they define :

$$F'^2 = (\Lambda^2 - m_i^2) \{\Lambda^2 - (k' - k)^2\}^{-1} \quad (89)$$

where m_i is the meson mass. For $(k' - k)^2 = m_i^2$ we find $F'^2 = 1$. Therefore a renormalized coupling constant $g_i'^2$ is defined by requiring $g_i'^2 F'^2 = g_i^2 F^2$. So :

$$g_i'^2 = \{\Lambda^2 (\Lambda^2 - m_i^2)^{-1}\} g_i^2 \quad (90)$$

From this form one observes that the renormalized values will be bigger than g_i^2 . This effect is especially clear for heavy mesons. For instance from table 12, page 147 one obtains :

$$\begin{array}{ll} g_\pi'^2 = 13.1 & \rho_\omega'^2 = 19.9 \\ g_\sigma'^2 = 5.3 & \rho_\delta'^2 = 8.3 \\ g_\eta'^2 = 7.1 & g_\phi'^2 = 13.3 \\ g_\rho'^2 = 2.1 & \end{array} \quad (91)$$

It is clear that the values depend critically on the cut-off procedure that is used. Therefore one can not take these too seriously.

From the various OBE-models that we discussed, one may notice that the results are not very sensitive to the specific method of unitarization that is chosen. Clearly the model allows enough freedom to parametrize important relativistic effects as the meson retardation.

A problem is that a satisfactory fit can not be made with a physical value for mass of the iso-scalar scalar. It is therefore reasonable to interpret the exchange of

this meson as an ad-hoc parametrization of neglected processes.

It seems impossible to get a quantitative fit for all phase parameters ($\chi^2 \sim 1$). Especially problems arise in obtaining a satisfactory fit to the 1D_2 and the 3D_2 phase-shifts (ref. 108). This is another indication that the OBE-assumptions form too rough an approximation.

However in the OBE-models one is able to reproduce the empirical phase parameters with an accuracy comparable with the best phenomenological potentials. The number of parameters involved is about 10, whereas for phenomenological potentials we need 30 to 40. Furthermore these parameters can, at least in principle, be obtained from other processes than nucleon-nucleon scattering.

One may therefore conclude that although it is not likely that the OBE-processes form the complete theory of the nucleon-nucleon interaction, they constitute a dominant component.

table_7 : masses of low energy mesons
from the particle data group (ref.92)

<u>meson</u>	<u>I(J^P)</u>	<u>mass(MeV.)</u>	<u>width(MeV.)</u>
π^\pm	$1(0^-)$	139.57	0.0
π^0		134.96	7.95 ± 0.55 eV.
η	$0(0^-)$	548.8 ± 0.6	0.85 ± 0.12 KeV.
$\rho(770)$	$1(1^-)$	773 ± 3	152 ± 3
$\omega(783)$	$0(1^-)$	782.7 ± 0.3	10.0 ± 0.4
$\eta'(958)$	$0(0^-)$	957.6 ± 0.3	<1
$\delta(970)$	$1(0^+)$	976 ± 10	50 ± 20
$S^*(993)$	$0(0^+)$	993 ± 5	40 ± 8
$\phi(1020)$	$0(1^-)$	1019.7 ± 0.3	4.1 ± 0.2
$\epsilon(1200)$	$0(0^+)$	1100~1300	600

Tables 8 - 13 :
Tables of meson masses and coupling constants used in the various models. The values between parentheses are prede-
termined, the other ones are adjusted. f/g denotes the
ratio of the direct and derivative coupling constants.
The names of the mesons in the tables are those used in
the model under consideration.

table_8 : SUWY (ref. 78)

<u>meson</u>	<u>I(J^P)</u>	<u>mass(MeV.)</u>	<u>g²</u>	<u>f/g</u>
π	$1(0^-)$	(140)	14.4	
S	$0(0^+)$	600	13.1	
ρ	$1(1^-)$	(750)	1.62	1.45
ω	$0(1^-)$	(750)	10.95	0.19

table_9 : Bryan-Scott (model III) (ref. 83)

<u>meson</u>	<u>I (J^P)</u>	<u>mass (MeV.)</u>	<u>g²</u>	<u>f/g</u>
π	$1(0^-)$	(138.7)	12.55	
η	$0(0^-)$	(548.7)	2.60	
σ_0	$0(0^+)$	550	8.19	
σ_1	$1(0^+)$	600	1.65	
ρ	$1(1^-)$	(763)	1.81	1.13
ω	$0(1^-)$	(782.8)	17.26	0.0

table_10 : Ueda-Green (model I) (ref. 84)

<u>meson</u>	<u>I (J^P)</u>	<u>mass (MeV.)</u>	<u>g²</u>	<u>f/g</u>
π	$1(0^-)$	(138.7)	14.01	
σ	$0(0^+)$	(416.1)	1.96	
η	$0(0^-)$	(548.7)	2.73	
ρ	$1(1^-)$	(763.0)	0.78	4.76
ω	$0(1^-)$	(782.8)	8.02	0.0
π_ν	$1(0^+)$	(1016.0)	4.11	
η_ν	$0(0^+)$	(1070.0)	4.44	

table_11 : Erkelenz et. al. (ref. 85)

<u>meson</u>	<u>I (J^P)</u>	<u>mass (MeV.)</u>	<u>g²</u>	<u>f/g</u>
π	$1(0^-)$	(140)	13.0	
σ_0	$0(0^+)$	500	4.7	
η	$0(0^-)$	(548)	6.0	
δ	$1(0^+)$	700	4.8	
ρ	$1(1^-)$	(780)	3.073	2.13
ω	$0(1^-)$	(780)	15.327	(0.0)

table_12 : Holinde and Machleidt (ref.97)

<u>meson</u>	<u>I (J^P)</u>	<u>mass (MeV.)</u>	<u>g²</u>	<u>f/g</u>
π	$1(0^-)$	138.5	13.0	
σ_0	$0(0^+)$	500.0	4.63	
η	$0(0^-)$	548.5	6.0	
ρ	$1(0^+)$	763.0	1.5	3.5
ω	$0(1^-)$	782.8	14.0	(0.0)
δ	$1(0^+)$	960.0	4.74	
Φ	$0(1^-)$	1020.0	7.0	(0.0)

table_13 : Scotti and Wong (ref. 107)

<u>meson</u>	<u>I (J^P)</u>	<u>mass (MeV.)</u>	<u>g²</u>	<u>f/g</u>
π	$1(0^-)$	(135)	(14)	
σ	$0(0^+)$	437	3.05	
η	$0(0^-)$	(548)	12.1	
ρ	$1(1^-)$	591	1.27	3.0
ω	$0(1^-)$	780	2.77	(0.0)
Φ	$0(1^-)$	1020	2.26	(0.0)

Appendix Scattering Amplitude and Transition Matrices

In the literature one may encounter various forms of the relation between the scattering amplitude $M_{i \rightarrow f}$ defined by :

$$\frac{d\sigma}{d\Omega} = |M_{i \rightarrow f}|^2 \quad (1)$$

and the transition matrix $C_{i \rightarrow f}$, defined with respect to the scattering matrix $S_{i \rightarrow f}$ by :

$$S_{i \rightarrow f} = \delta_{if} + i(2\pi)^4 \delta(k_f - k_i) C_{i \rightarrow f} \quad (2)$$

Wong (ref. 43) uses the relation :

$$M_{i \rightarrow f} = \frac{M^2}{4\pi E} C_{i \rightarrow f} \quad (3)$$

Sawada et. al. (ref. 78) employ :

$$M_{i \rightarrow f} = \frac{E}{4\pi} C_{i \rightarrow f} \quad (4)$$

While in the textbook of Nishijima (ref. 110) one finds :

$$M_{i \rightarrow f} = \frac{1}{16\pi} \frac{1}{E} C_{i \rightarrow f} \quad (5)$$

Some more forms may be found differing in factors (2π) or M . We shall confine ourselves to (3), (4) and (5), and show how these can be accounted for.

The differential cross-section is the probability per unit space-time that an incoming particle is scattered by a target nucleon into a solid angle $d\Omega$. (We suppress the internal degrees of freedom which are immaterial to

this discussion.) The probability P that the state of the two nucleon system changes in the collision process, is :

$$P = \int d\vec{k}_f^1 d\vec{k}_f^2 e(E_f^1) e(E_f^2) |S_{i \rightarrow f}|^2 \quad f \neq i \quad (6)$$

$d\vec{k}_f e(E_f)$ is a one particle phase-space element. There is no need to specify $e(E_f)$ at this stage of the discussion. Upon substitution of (2) into (6) and using :

$$(2\pi)^4 \delta(k_f - k_i) = VT \quad , \text{ total space-time} \quad (7)$$

we find the total transition probability per space-time :

$$\frac{P}{VT} = \int \int d\vec{k}_f^1 d\vec{k}_f^2 e(E_f^1) e(E_f^2) (2\pi)^4 \delta(k_f - k_i) |C_{i \rightarrow f}|^2 \quad (8)$$

It is convenient to transform to the cm-system. After a little manipulation we find from (8) :

$$\frac{d}{d\Omega} \frac{P}{VT} = (2\pi)^4 \frac{1}{2} |\vec{k}| E e^2(E) |C_{i \rightarrow f}|^2 \quad (9)$$

where E and $|\vec{k}|$ are the cm-energy and the momentum magnitude for a nucleon. (We are not justified to use (9) for $\Omega=0$.) $\frac{d\sigma}{d\Omega}$ is obtained by calculating $\frac{d}{d\Omega} \frac{P}{VT}$ for one incoming nucleon and one target nucleon. To this end we should divide (9) by the flux-density F.D., which is discussed below :

$$\frac{d}{d\Omega} \sigma(\Omega) = \frac{1}{\text{F.D.}} \frac{d}{d\Omega} \frac{P}{VT} \quad (10)$$

Flux-density

We consider a complete set of one nucleon states $\{|\vec{k}\rangle\}$

$$\int d\vec{k} \, e(E) |\vec{k}\rangle \langle \vec{k}| = I \quad (11)$$

Using this completeness requirement, we see that :

$$\langle \vec{k}' | \vec{k} \rangle = \frac{1}{e(E)} \delta(\vec{k} - \vec{k}') \quad (12)$$

We interpret $\frac{1}{V} \langle \vec{k} | \vec{k} \rangle$ as the particle density :

$$\frac{1}{V} \langle \vec{k} | \vec{k} \rangle = \frac{1}{(2\pi)^3} \frac{1}{e(E)} \quad (13)$$

The flux-density is the product of the densities of the incoming and target particles and their relative velocity v :

$$F.D. = \frac{1}{(2\pi)^6} \frac{1}{e(E^1)e(E^2)} |v| \quad (14)$$

For the cm-system one obtains therefore :

$$F.D. = \frac{1}{(2\pi)^6} \frac{2|\vec{p}|}{e^2(E)E} \quad (15)$$

By substitution of (15) and (8) into (9), we find :

$$\frac{d}{d\Omega} \sigma(\Omega) = \frac{(2\pi)^{10}}{4} E^2 e^4(E) |C_{i \rightarrow f}|^2 \quad (16)$$

We consider now the form of the phase-space element $e(E)$. Wong uses :

$$e(E) \rightarrow \frac{1}{(2\pi)^3} \frac{M}{E} \quad (17)$$

The factor $\frac{M}{E}$ has a simple interpretation. If one considers box quantization, it represents the Lorentz contraction of the box. By substitution of (17) into (16),

we obtain the relation (3).

The result (4) of Sawada et. al. can be reproduced by using the non-relativistic phase-space element :

$$e(E) \rightarrow \frac{1}{(2\pi)^3} \quad (18)$$

The relation used by Nishijima is obtained by choosing the form :

$$e(E) = \frac{1}{(2\pi)^3} \frac{1}{2E} \quad (19)$$

This form leads to the covariant phase-space element :

$$\frac{1}{(2\pi)^3} \frac{d\vec{p}}{2E} = \frac{d^4p}{(2\pi)^3} \delta(p-M) \theta(p_0) \quad (20)$$

Of course a consequent use of the two covariant forms (17) and (20) leads to the same predictions about $\frac{d\sigma}{d\Omega}$ from a model of the interaction. The phase-space element used by Sawada et. al. is non-covariant and the use of (4) is justified for $\frac{M}{E} \approx 1$ only. In the case of a laboratory kinetic energy $E_{\text{lab}} = 350 \text{ MeV.}$, we find $M/E \approx 0.84$.

CHAPTER IV

BEYOND THE ONE BOSON EXCHANGE

1) Introduction

The OBE-models that we discussed in the previous chapter, are not satisfactory in the following four respects. First a cut-off procedure has to be introduced if we want to describe the lower partial waves. This introduces phenomenological parameters on which the values of the coupling constants for the heavier mesons depend critically. Secondly a fictitious scalar meson appears to be indispensable. Clearly the values for the well-established mesons will depend on the mass and coupling constant of the scalar meson σ . Therefore one can not really compare the OBE-coupling constants with the values from other sources. It has been stated often that the form factor and the σ parametrize the higher order effects that are not treated explicitly in the OBE-models. Thirdly, this parametrization does not lead to the desired quantitative fit with the data. Finally, the semi phenomenological character of the OBE-description introduces uncertainties in many-body applications.

There are therefore both fundamental and practical arguments to consider higher order effects explicitly. The most important of these is the exchange of two uncorrelated pions. In section two we consider potentials that are derived from the fourth order Feynman graphs. In these models one encounters problems concerning the form of the interaction Hamiltonian. The ps-pv (pseudo scalar pseudo vector = derivative) coupling leads to divergences, while the unmodified ps-ps (pseudo scalar pseudo scalar = direct)

coupling is not justified. The ps-ps coupling theory overestimates the contributions due to virtual nucleon-antinucleon pairs.

It is shown in section three that the introduction of a (fictitious) scalar meson σ may reduce the problems that one encounters in a fourth order calculation of the two pion exchange. But this is still not satisfactory for obvious reasons. In the second part of section three the question is considered to what extent one can replace the σ in an OBE-description, if one includes the effects of $\Delta(1232)$ formations in the intermediate state. The conclusion is that, at least for the long range attraction, the $\Delta(1232)$ can replace the σ . But for the more inner lying region, a semi phenomenological description in terms of the σ and a form factor is needed.

The problems connected with the fourth order calculation of the two pion exchange can be circumvented if dispersion relations are used. This method has furthermore the advantage that interactions between the two pions and the pion-nucleon rescattering effects can be incorporated in a natural way. This method is discussed in section four.

The introduction of the two pion exchange effect may render the σ exchange superfluous, but a cut-off factor still has to be used. A somewhat less arbitrary choice of a formfactor is however possible than the one discussed in chapter III, §3. The so called eikonal form factor is based on the multiple exchange of neutral vector mesons. The use of this formfactor in OBE-potentials leads to more acceptable values for the coupling constants. This form factor is the topic of section five.

In section six the question is briefly considered which other higher order processes are relevant for the nucleon-nucleon interaction problem. Furthermore a few concluding remarks are made.

2) Two Pion Exchange : the Fourth Order Potentials

Before the discovery of the pion resonances a lot of theoretical effort was invested in establishing the two pion exchange effect. An important question in these investigations concerned the form of the interaction Hamiltonian. If we go beyond the Born approximation, the ps-ps - and ps-pv couplings do not have the same on-shell behaviour. A practical difference between the two coupling modes is that the ps-ps coupling theory can be renormalized using the Dyson technique, while no renormalization scheme is known in case of a ps-pv coupling. Another difference can be seen from the appearance of γ^5 in the ps-ps coupling Lagrangian. γ^5 will couple negative - and positive energy components, so one expects significant contributions from intermediate nucleon-antinucleon pairs. An interesting relation between the two coupling modes was discovered by Dyson (ref. 114). To second order in the coupling constant, the ps-ps coupling term can be written essentially as the sum of the ps-pv coupling term and a term quadratic in the coupling constant and the pion field. This quadratic term arises from the contributions of the intermediate nucleon-antinucleon pairs. Due to the large value of the pion-nucleon coupling constant, one expects therefore a large spin independent contribution from this term. However experimentally no strong S-state interaction was observed in pion-nucleon scattering. This result indicates that the contribution from the formation of an intermediate nucleon-antinucleon pair is small. This fact does not necessarily imply that one should use ps-pv coupling. Drell and Henley (ref. 115) showed that the primary effect of the interaction terms coming from the difference between ps-ps - and ps-pv coupling consists of a short range repulsion. Potentials of this sort

will give a large contribution to the S-wave interaction in the Born approximation, but in an exact theory the effects may be small. This analogy suggests that if one takes higher order contributions into account, one may obtain the observed 'damping' of the effects of the virtual nucleon pair. Attempts in this direction, for instance by Hamada and Shono (ref. 116), indicate that damping may be obtained from a selective treatment of radiative corrections. One expects this damping of the virtual nucleon pairs in pion-nucleon scattering also to be of importance for the two pion exchange contribution to the nucleon-nucleon interaction (compare figure 30a and 30b, 30c).

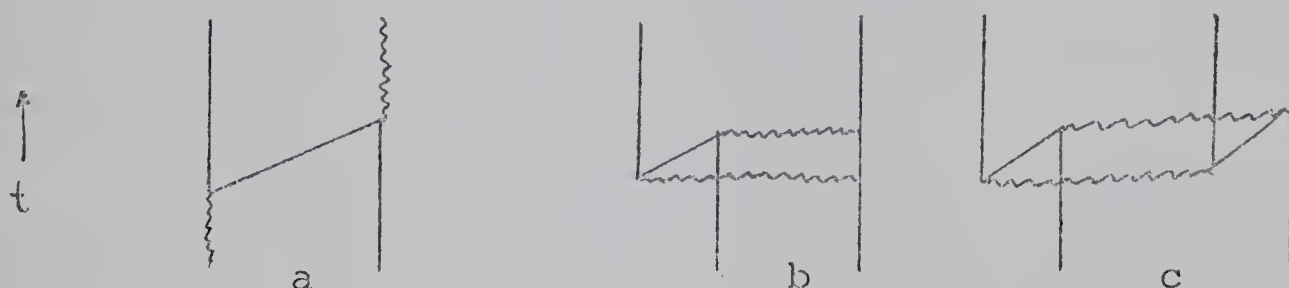


figure 30 : virtual nucleon pairs in πN scattering (fig. 30a) and NN scattering (fig. 30b, 30c)

In a radical approach one might assume that the difference between ps-ps coupling with pair suppression and ps-pv coupling is negligible. This attitude was taken by Taketani et. al. (ref. 117) in 1952. They calculated fourth order potentials corresponding to the box diagram and the crossed diagram, using ps-pv coupling. In this calculation unrenormalisable divergences appear. They noticed that in the static limit these divergences vanish. In this limit one assumes non-relativistic nucleons and a vanishing pion-nucleon mass ratio. It was hoped that for the outer range of the interaction, this potential supplemented with the OPE-contribution might prove a reasonable description. The inner region was treated entirely phenomenologically.

A similar potential was constructed by Brückner and Watson (ref. 118). A qualitative agreement with the successful phenomenological Hamada-Johnston potential is obtained in these approaches.

Nevertheless the static limit in these calculations is clearly an undesirable feature. Charap et. al. (ref. 119) tried to avoid this limit. However in their model they used the approximation of non-relativistic external momenta for the one pion exchange. As was noticed by Partovi and Lomon (ref. 96), the calculation of the fourth order diagram involves the iteration of the one pion exchange contribution over all momenta, and therefore a non-relativistic approximation to these amounts to a partially static approximation. They used $\pi\pi$ coupling and neglected contributions from virtual nucleon pairs.

The problem they encountered was the appearance of a divergency which was removed ad-hoc by a subtraction of a zero range and an infinite strength potential. Cottingham and Vinh Mau (ref. 120) used this formalism to calculate a two pion exchange potential. They obtained too strong an iso-singlet central potential. Partovi and Lomon (ref. 96) pointed out that this can be traced to the subtraction.

It is interesting that the $\pi\pi$ coupling leads to a convergent contribution from the box and crossed diagrams. This result indicates that the virtual nucleon pair provides a cut-off. We can expect contributions from the virtual nucleon pairs for cm-momenta of the order of the nucleon mass or more. Therefore in order to obtain convergence, a correct relativistic treatment of the high momentum part of the integration is quite important. The convergence of the fourth order terms and the renormalizability are important practical arguments in favour of the use of the $\pi\pi$ coupling theory. Partovi and Lomon (ref. 96)

constructed a potential based on the exchange of two uncorrelated pions, besides the exchange of one pion, the ω , ρ and η . The formalism that they use is based on the Blanckenbecler-Sugar reduction of the Bethe-Salpeter equation (ref. 113). To this end they extended the Blanckenbecler-Sugar equation to the case of spin- $\frac{1}{2}$ particles. The Bethe-Salpeter equation can be written symbolically in the form :

$$C = U + UgC \quad (1)$$

$$U = K + K(G-g)U \quad (2)$$

In this form K is the Bethe-Salpeter kernel; G the relativistic propagator and g a non-relativistic reduction of it. U is the potential. By elimination of U one easily sees that (1) and (2) are indeed equivalent to the Bethe-Salpeter equation. The kernel K can be written as an expansion in powers of the coupling constant :

$$K = K^{(2)} + K^{(4)} + \dots \quad (3)$$

Accordingly we can expand U :

$$U = U^{(2)} + U^{(4)} + \dots \quad (4)$$

by substitution of (3) and (4) in (2), we obtain :

$$U^{(2)} = K^{(2)} \quad (5)$$

$$U^{(4)} = K^{(4)} + K^{(2)}GK^{(2)} - K^{(2)}gK^{(2)} \quad (6)$$

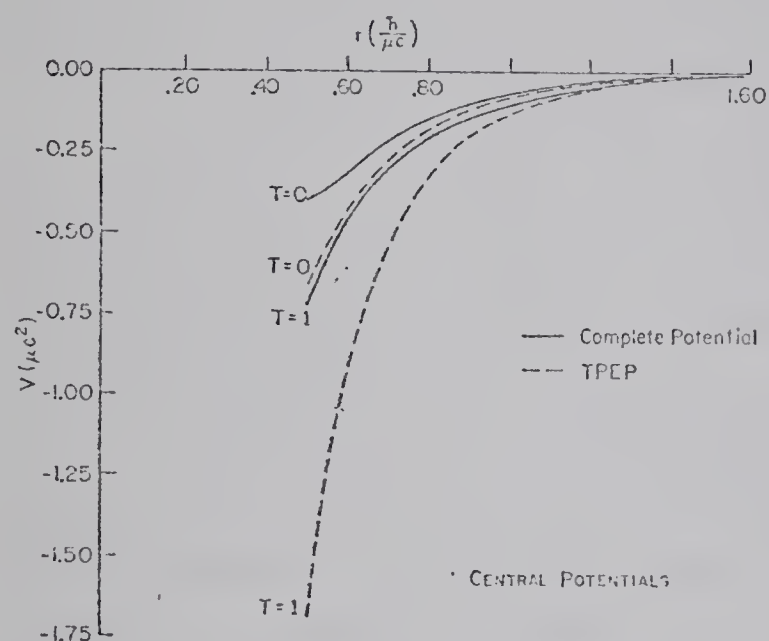
$K^{(2)}$ comes from the one meson exchange. A restriction to $U^{(2)}$ yields the OBE-description discussed in chapter III, §4. $U^{(4)}$ is the fourth order potential. $K^{(2)}GK^{(2)}$ clearly

represents the box diagram, therefore $K^{(4)}$ gives the remaining fourth order contribution which comes from the crossed diagram (the other fourth order diagrams contribute to mass and vertex renormalizations). The term $K^{(2)}gK^{(2)}$ is the iterated OPEP, which should be subtracted to avoid double counting. The non-relativistic two nucleon propagator g is required to have the same cut structure as G in the physical region. This requirement does not specify the choice of g uniquely. However the significance of the Blanckenkecler-Sugar choice is that it gives a reduction to the non-relativistic Lippmann-Schwinger formula. Of interest in this connection is the observation made by Jackson et. al. (ref. 121), that the various three-dimensional reductions of the Bethe-Salpeter equation lead to significantly different results in the framework of the OBE-restrictions. However if higher order contributions, such as the two pion exchange, are included, the differences between the various choices for g become much less important. They concluded that really significant is only that g contains positive energy projection operators. The reason is that otherwise the cancellation between the large negative energy contributions, which are present in the box and crossed diagram for ps-ps coupling, will be disturbed. This will worsen the convergence of (4).

The calculation of the fourth order contributions is not a simple matter, since it involves a four-dimensional numerical integration, but doable. The identity of the cut structure of G and g in the physical region causes cancellations between $K^{(2)}gK^{(2)}$ and the terms $K^{(4)}$ and $K^{(2)}GK^{(2)}$. Furthermore ps-ps coupling was used, so the integrals converge. Partovi and Lomon made an expansion in powers of $\frac{|\vec{Q}|}{M}$, where $\vec{Q}=\vec{k}+\vec{k}'$. For simplicity in the transformation to the coordinate representation, only terms of

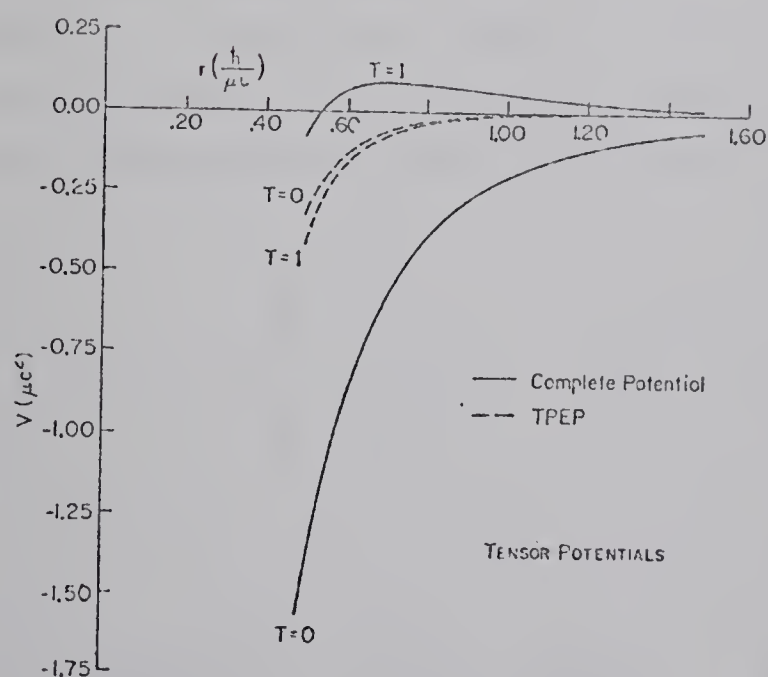
first order in $\frac{|\vec{Q}|}{M}$ were retained. One can not expect that the adiabatic approximation is justified in the inner region of the interaction. Therefore their consideration was confined to the range outside 0.8fm. This adiabatic approximation was also made for the OBE-potentials. So a potential of the Wigner-Eisenbud form is obtained. An interesting peculiarity in their approach is that the mass differences between the charged π^\pm and neutral pion are accounted for. This implies that charge independence does not hold for their model (of course charge symmetry remains to be valid). Values for coupling constants and meson masses are all predetermined. The not too well known coupling constants for the η and ω -meson are estimated on the basis of SU(3) and predictions from Oakes and Sakurai (ref.122). No attempt was made to calculate phase parameters. Instead they compared the potential for $r > 0.8\text{fm}$ with other known potentials. A reasonable agreement was found with the Hamada-Johnston potential except for the spin-orbit potential. This is not surprising as no quadratic spin-orbit term was included. To exhibit the relative importance of the TPE (two pion exchange)-contribution, the TPE is shown separately from the total potential. It turned out that TPE is a major contribution except for the tensor forces which are largely determined by the OPE. Another interesting observation was that the TPE-potential does not resemble an iso-scalar scalar exchange in the central and spin-orbit potentials, although it does for the tensor and spin-spin parts. This fact is considered as an indication for the need of employing iso-vector scalar mesons in OBE-models. The size of the TPE-contribution clarifies the often used unreasonable large values for the coupling constants of the σ and ω -mesons in the OBE-models. Figures 31 and 32 show the central π - and tensor potentials of the complete potential and the TPE-contribution to it. In the Partovi-

Lomon model no nucleon iso-bars are included, although in their paper the necessary formalism to this end is developed. A problem in their model may be due to double counting as both the uncorrelated two pion exchange and the ρ -meson are included. The ρ is a wide resonance of the 2π -system. The justification that the authors give is that the mass of this resonance is rather high, so that it is well-separated from the two pion continuum.



figure_31 :

central potential in the Partovi-Lomon model and the TPE-contribution to it



figure_32 :

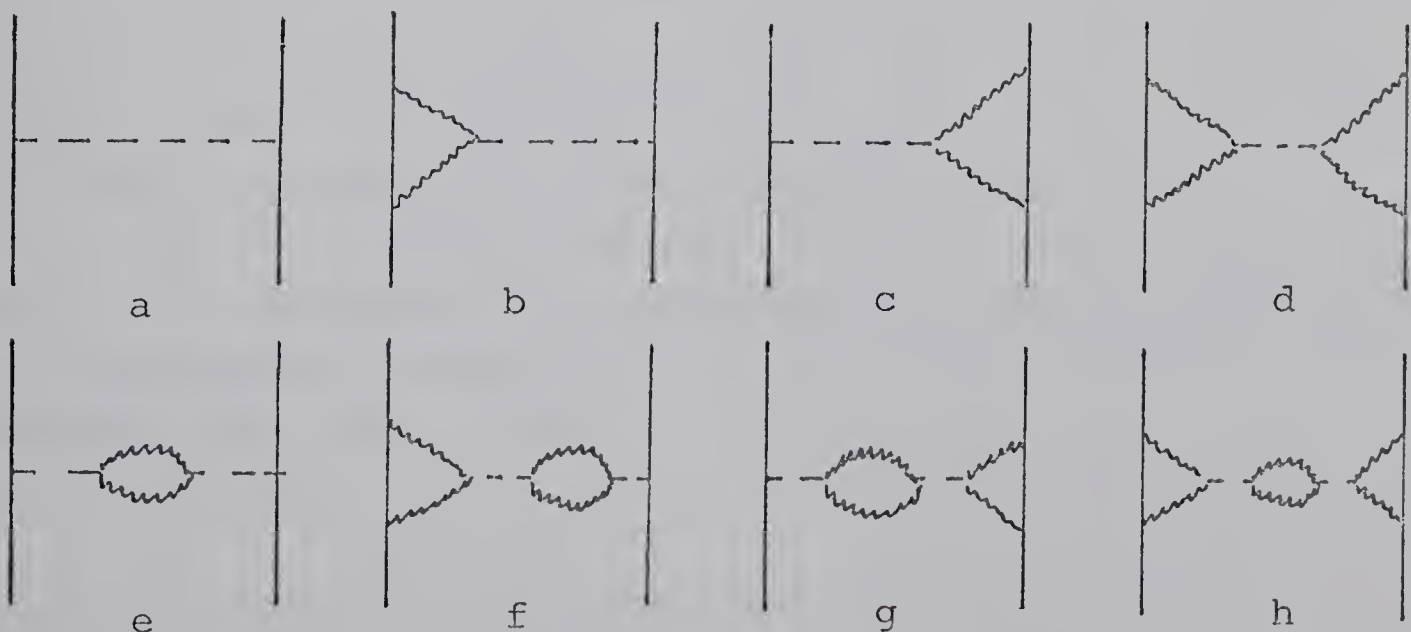
tensor potential in the Partovi-Lomon model and the TPE-contribution to it

The iso-triplet potentials are actually the $T_3 = \pm 1$ components.

3) The Iso-Scalar Scalar

In the H.M.Partovi-Lomon potential the ps-ps coupling was used for the pion-nucleon interaction. The most important reason for doing so, is that it allows convenient calculations. The integrals for the box and crossed diagrams are convergent and the theory is renormalizable. However, as we discussed in the beginning of this chapter, the ps-ps coupling theory predicts much too large pion-nucleon scattering lengths in the lowest order.

A way out is the use of a ps-ps coupling theory, but added to it a pair suppression mechanism. This attitude was taken by F.Partovi and Lomon (ref. 133). They considered the H.M.Partovi-Lomon potential (ref. 96) and included an iso-scalar scalar σ to provide the pair suppression. The σ was introduced originally by Schwinger (ref. 134) in connection with axial vector currents and chiral symmetry. Gell-mann and Levy (ref. 135) used chiral symmetry arguments to show that by introducing the σ , cancellations can occur between the various terms in the pion-nucleon scattering length, so that the required small value is obtained. More details on the so called σ -model can be found in the references 136 and 137. Lomon and Partovi considered the following diagrams :



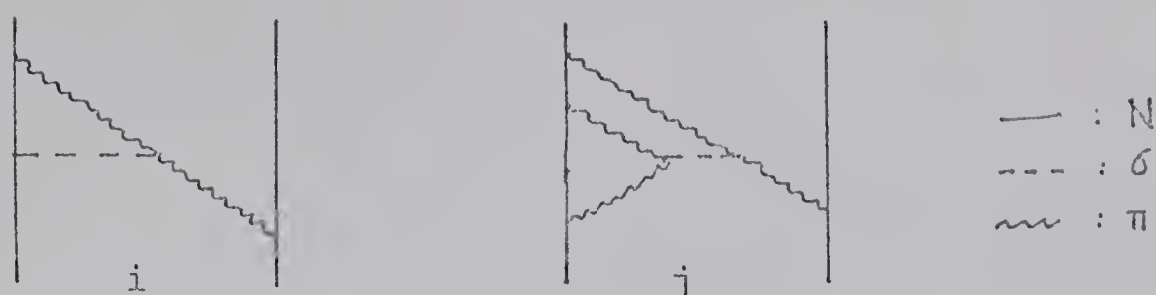


figure 33 : diagrams, included in the F.Partovi-Lomon model

The contributions of the diagrams a, b, c and d were calculated by adding these to the kernel of the modified Bethe-Salpeter equation. The effects of the bubble diagrams such as e, f, g and h, is to change the σ -propagator from that of a stable particle to that of an unstable particle with mass M_σ and width Γ_σ . The diagrams i and j can be interpreted as vertex corrections to the one pion exchange, their dominant effect is included by applying a coupling-constant renormalization.

The coupling constants $g_{\sigma N}$ and $g_{\pi\sigma}$ can be related to the σ -mass and σ -width by equations that follow from the chiral invariance and soft pion dynamics. These considerations also lead to the relation $m_\sigma = m_\rho = 765$ MeV.. Cancellations with the pair contributions to the potential were obtained by choosing $\Gamma_\sigma = 610$ MeV.. This cancellation is illustrated for the central potential in figure 34 on the next page. The pair contributions to the potential shown in this figure, are calculated by Klein (ref. 138), who used a Tamm-Dancoff formalism. Partovi and Lomon concluded that the inclusion of the σ decreased the differences with the Hamada-Johnston potential. The results are shown in figure 35, page 163, for the central potential.

This approach is, however, not very satisfactory, as the σ is introduced in a rather ad-hoc fashion. There is no experimental evidence for an iso-scalar scalar of the assumed mass and width.

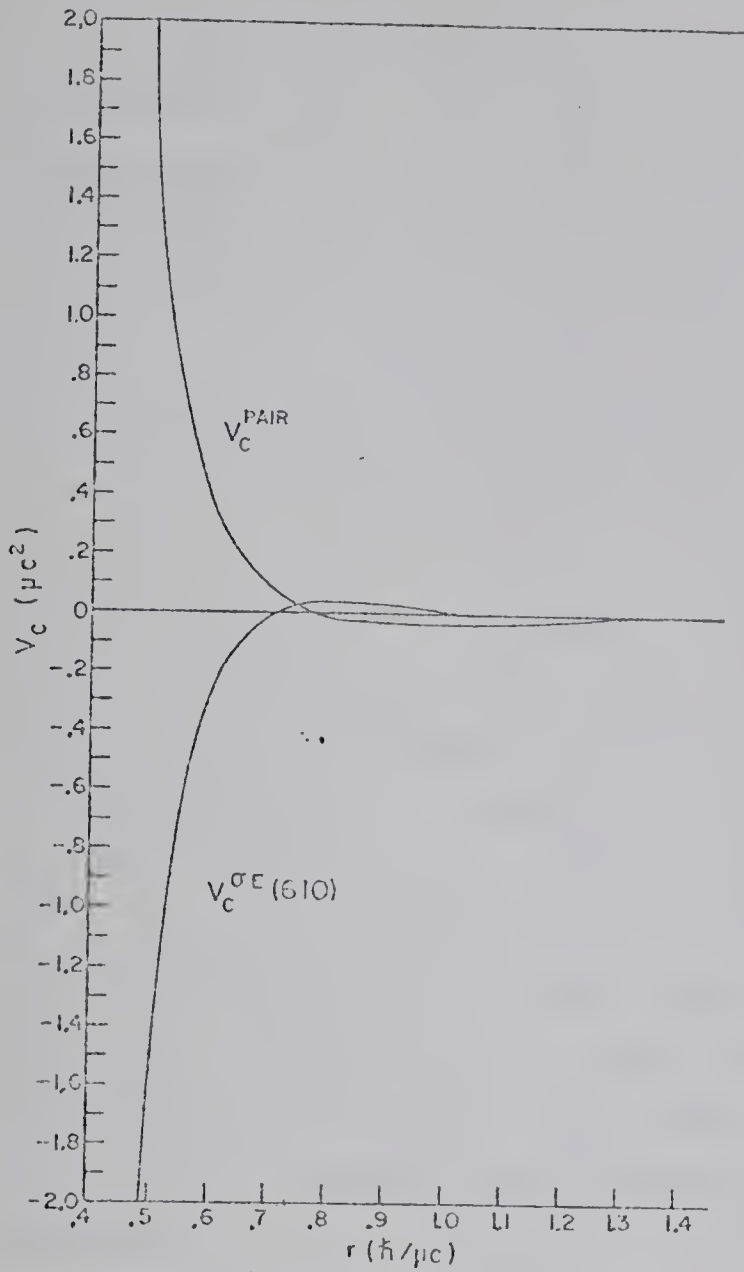
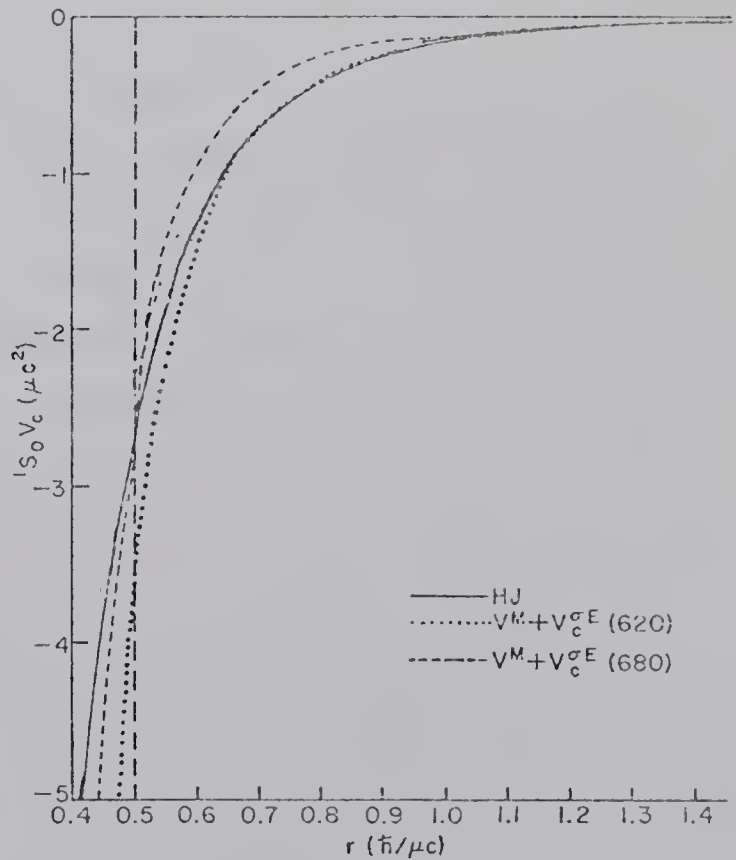


figure 34 :

the σ -exchange central potential for $\Gamma = 610$ MeV. compared to the (non-relativistic) one - and two pair contributions to the central NW potential (ref. 138)

figure 35 :

the central potential in the $1S_0$ state of the F.Partovi-Lomon for $\Gamma = 610$ MeV. and $\Gamma = 680$ MeV., compared to the Hamada-Johnston potential (ref. 34)



An interesting observation in this respect was made by Chew et. al. (ref. 123) in their classical article on dispersion relations for pion-nucleon scattering. It was well-known that the S-wave scattering lengths are badly overestimated in a model consisting of a N-pole term only. Chew et. al. noticed that a zero-width $\Delta(1236)^*$ pole in the dispersion relations has an effect of the same order of magnitude as the effect of the N-pole term and tends to cancel it, as required. It is therefore tempting to attribute the role of pair suppression to the nucleon iso-bars, instead of to the more or less fictitious σ . All realistic OBE-models include the exchange of the σ . It would surely be interesting to study to what extent the introduction of the $\Delta(1236)$ in the formalism can replace the σ . An attempt in this direction was made by Green and Haapakoski (ref. 139). However their study concerned the 1S_0 partial wave only. Another restriction was that just the elastic NN and the inelastic $N\Delta$ -channel were considered, thereby neglecting contributions from the $\Delta\Delta$ -channel.

A further simplification was that the potential in the $N\Delta$ -channel was taken to be the same as the NN-channel potential. For the diagonal and transition potentials forms were chosen, suggested by the π , η , ω and ρ -exchange, in which two free parameters were introduced. It appeared possible to obtain a reasonable fit to the 1S_0 phase-shift. This was achieved without introducing a σ -meson. The calculations of Green and Haapakoski were extended by Smith and Pandharipande (ref. 140) to phase-shifts for $L \leq 2$. They studied the effects of the diagrams shown in figure 36, for the special case where all external nucleons are at rest (see next page).

*) In more recent tables one denotes this iso-bar as $\Delta(1232)$.

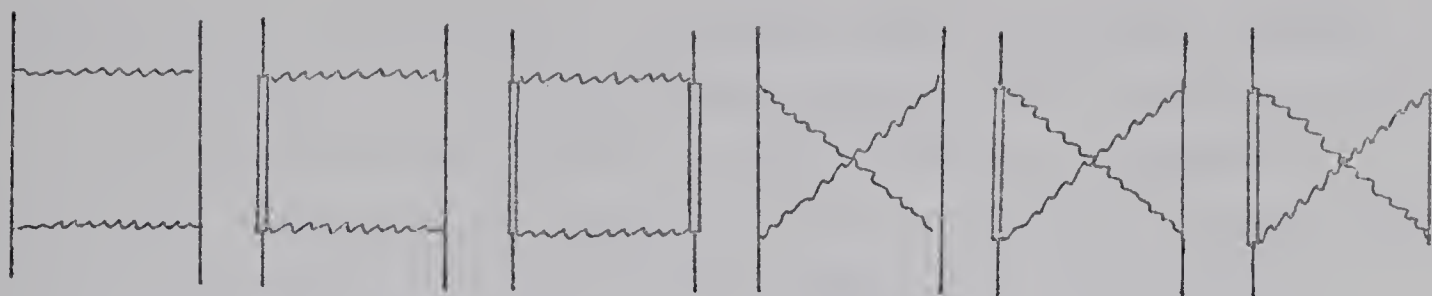



figure 36 : diagrams, included
in the model of Smith
and Pandharipande
(ref. 140)

— : N
 : Δ
 ~ : π

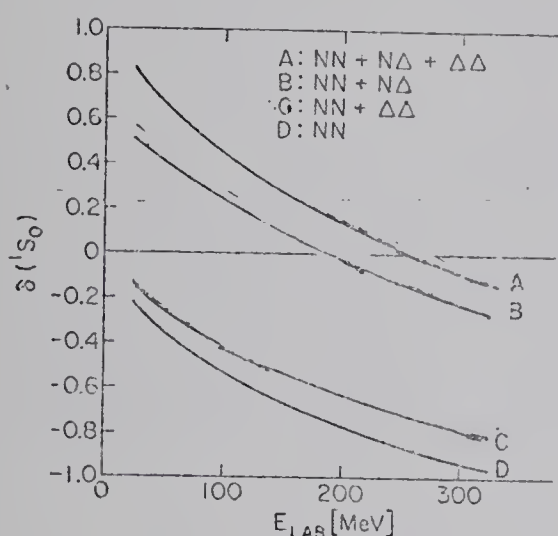


figure 37 :

results for the 1S_0 phase-
shift using various combinations
of the channel NN , NΔ , ΔΔ

Non-relativistic approximations were used and the short range repulsion was described phenomenologically. Figure 37 shows their results for the various combinations of the channels NN , NΔ and ΔΔ . One notices that the introduction of NΔ and ΔΔ gives an extra attraction and that the effects of the ΔΔ-channel, although smaller than the effects of the NΔ-channel, are significant. A problem in the model was that the D-state phase-shifts were predicted too attractive. In view of the non-relativistic approximations and the phenomenological description of the repulsion, one can not attribute too much weight to their results. Holinde and Machleidt (ref.141) calculated a momentum space nucleon-nucleon potential in the OBE-framework in which the intermediate range attraction, resulting from the inelastic NΔ and ΔΔ-channels, were incorporated. No non-

relativistic restriction was made. Their approach however also contained a few simplifying assumptions, which we illustrate by considering the coupled channel Lippmann-Schwinger equations in terms of the K-operator. Using the symbolic notation of Holinde and Machleidt (ref. 141), these can be written as :

$$K_{\alpha}^{\beta} = V_{\alpha}^{\beta} + \sum_{\gamma} V_{\gamma}^{\beta} G_{\gamma}^{\gamma} K_{\alpha}^{\gamma} \quad (7)$$

The parameters α , β and γ denote the channels, and G_{γ}^{γ} is the propagator for the channel γ . The subscript and superscripts denote the incoming - and outgoing state respectively. Holinde and Machleidt assume firstly that the transitions between the $N\Delta$ and $\Delta\Delta$ -channels may be neglected if we are interested in the NN -channel only, and secondly they neglect the interactions in the $N\Delta$ - and $\Delta\Delta$ -channels. These approximations imply respectively :

$$V_3^2 = V_2^3 = 0 \quad (8a)$$

$$V_2^2 = V_3^3 = 0 \quad (8b)$$

where we denote by 1 , 2 and 3 respectively the channels NN , $N\Delta$ and $\Delta\Delta$. The transition potentials are iterated to second order. From (8a) and (8b) follows :

$$K_1^{\delta} = V_1^{\delta} + V_1^{\delta} G_1^1 K_1^1 \quad \text{where } \delta \in \{2, 3\} \quad (9)$$

Substitution of (9) in (7) for the NN -channel under the conditions (8a) and (8b) gives the convenient expression :

$$K_1^1 = V_{\text{eff}} + V_{\text{eff}} G_1^1 K_1^1 \quad (10a)$$

where :

$$V_{\text{eff}} \equiv V_1^1 + V_2^1 G_2^2 V_1^2 + V_3^1 G_3^3 V_1^3 \quad (10b)$$

The entire contribution of the Δ in this approach is included in the effective potential V_{eff} . In this procedure they only considered the effect of the Δ for the iterated pion exchange. The transition potentials V_1^1 and V_2^1 are determined from the $\pi N\Delta$ coupling Lagrangian :

$$\mathcal{L}_{\pi N\Delta} = \sqrt{4\pi} \frac{f_\pi^*}{m_\pi} \bar{\Psi} T \Psi^\mu \partial_\mu \phi + \text{h.c.} \quad (11)$$

Ψ and ϕ denote, as usual, the nucleon and pion fields. Ψ^μ denotes the field operator for the Δ . T is the iso-spin operator. Furthermore a dipole type cut-off factor was introduced. Explicit derivations and expressions of the transition potentials are given in references 141 and 142. Phase-shifts were calculated, using the contributions from their OBE-model (ref. 131). V_1^1 in the equations (10a) and (10b) was simply put equal to the OBE-potential. The calculated phase-shifts appeared very sensitive to the value of the cut-off mass. Figures 38, 39 and 40 on page 168 and 169, illustrate their results for the 1S_0 , 1D_2 and 3D_1 phase-shifts. These graphs show the results of the OBE-model, including the potential due to the σ -exchange, the results if the σ -exchange contribution is left out, and the results if instead of the σ -exchange the Δ -effects are added. Three values for the cut-off mass Λ were considered. One notices that for the 1S_0 phase-shift the Δ -contribution nearly replaces the contribution from the σ , for $\Lambda=850$ MeV.. However for the 1D_2 partial wave this choice gives too much attraction. This result holds for all higher partial waves, for which the total iso-spin $T=1$. It is interesting that for the $T=0$ phase-shifts the

attraction is much weaker, as is illustrated in figure 40 for the 3D_1 phase-shift. The interpretation of this result is simple as the $N\Delta$ -channel does not contribute in that case. From figure 40 one observes that the effects of the $\Delta\Delta$ -channel are important, especially for large values of the energy, which indicates that the attraction from the $\Delta\Delta$ -channel is short ranged. These results show that in order to obtain a good fit to the data, some contribution of the σ is necessary. This is not really a surprise as the iterated OPE-potential does include effects of the box diagrams, but not of the crossed diagrams with intermediate Δ -states. After establishing the qualitative effects of the Δ , Holinde and Machleidt made quantitative fits with the model to the data for three values of Λ . The values for the coupling constants that were found, depend critically on the value for the cut-off mass, and are therefore not very interesting. It is significant however that the optimum value for the σ -mass turned out to be 700 MeV. which is 200 MeV. larger than one finds without the Δ . This indicates that especially the short range attraction is not well-described by the Δ -effects. It should be mentioned that the introduction of the Δ appeared to give important effects in many-body calculations, that can not be accounted for by the σ (ref. 139 and 141).

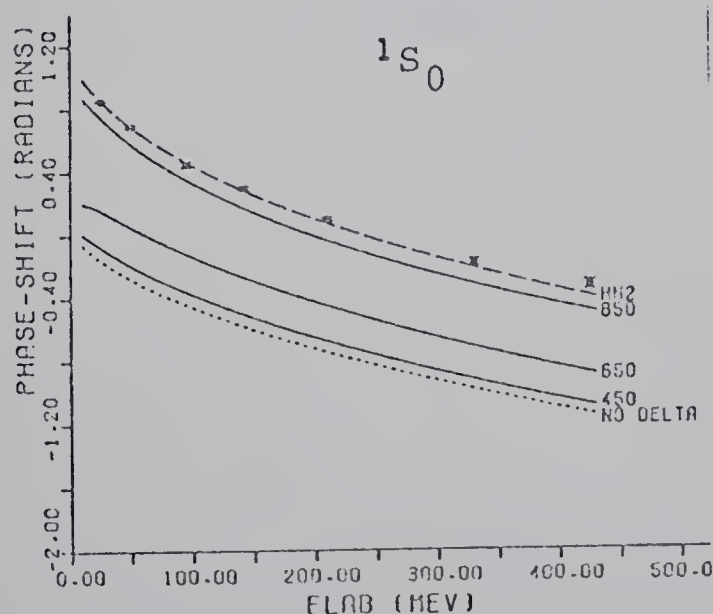


figure 38

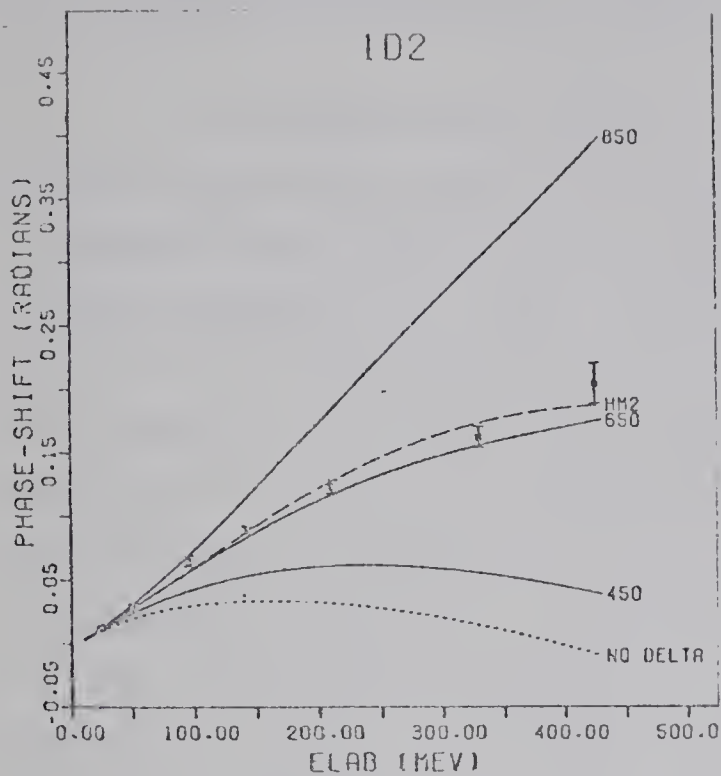


figure 39

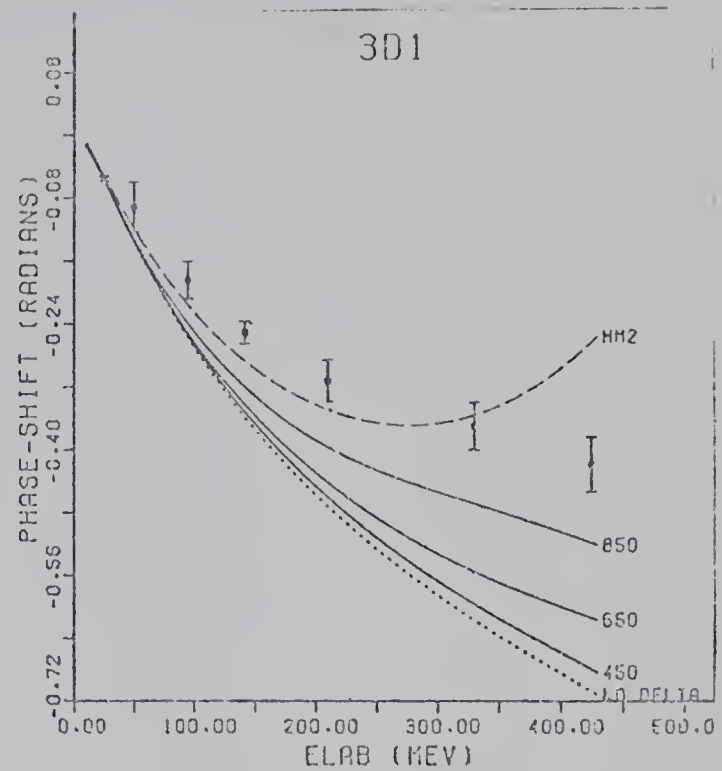


figure 40

Phase-shifts for the 1S_0 (fig. 38), 1D_2 (fig. 39) and 3D_1 (fig. 40) waves. The dashed curve shows the results of the OBE-model (ref. 131). For the dotted curve the σ -contribution is left out from this OBE-model. The solid curves show the results when the σ -contribution is replaced by the $N\Delta$ interaction for different values of the cut-off mass. Empirical values are taken from the Livermore analysis (ref. 98).

4) Two Pion Exchange : Dispersion Relations

A number of authors feels that the dispersion relation technique offers the more appropriate formalism to include the effects of nucleon iso-bars and π - π interactions. Especially the effects of large width resonances in the π - π system are incorporated in a natural way in the dispersion relation approach. Furthermore the problems inherent to the use of fourth order TPE-calculations are circumvented in the dispersion relation approach. In it independent empirical information, concerning $\pi N \rightarrow \pi N$ and $\pi \pi \rightarrow \bar{N} N$ scattering processes, is used to determine the TPE-effect. In the literature at least three procedures can be found in which dispersion relation techniques are developed for applications to the 2π -exchange effects in N-N interactions: by Goldberger et. al. (ref. 103), the Japanese School of Furuichi and collaborators (ref. 52), and by Amati, Leader and Vitale (ref. 130). The formulation by Amati et. al. probably enjoys most popularity. In 1971 Chemtob et. al. (ref. 109) proposed a hybrid formulation in which a potential model is obtained. The approach is based on the Blanckenbecler-Sugar equation. The one meson exchange contributions are calculated using the Lagrangian formalism, but the two pion exchange effect is determined using the dispersion relation formalism of Amati et. al.. Thereby some errors made by these authors, were corrected. In their model, besides the one - and two pion exchanges, the η and the ω are included. Both η and ω are narrow three pion exchange resonances and can be put in as elementary exchanges. We shall briefly review their method and the results that are obtained with it.

In order to make convenient comparisons with the literature possible, in the following we shall use the

conventions concerning normalization of Amati et. al. and Chemtob et. al.. The causal amplitude A can be written as a linear combination of a set of relativistically invariant operators, whose coefficients are scalar functions of the Mandelstam variables. A frequently used set of invariants is :

$$\begin{aligned} P_1 &\equiv I_1 I_2 \quad ; \quad P_2 \equiv (i\gamma_1 \cdot P I_2 + I_1 i\gamma_2 \cdot N) \quad ; \\ P_3 &\equiv (i\gamma_1 \cdot P)(i\gamma_2 \cdot N) \quad ; \quad P_4 \equiv \gamma_1 \cdot \gamma_2 \quad ; \quad P_5 \equiv \gamma_1^5 \gamma_2^5 \end{aligned} \quad (12)$$

where $N = \frac{1}{2} (k_1 + k'_1)$ and $P = \frac{1}{2} (k_2 + k'_2)$

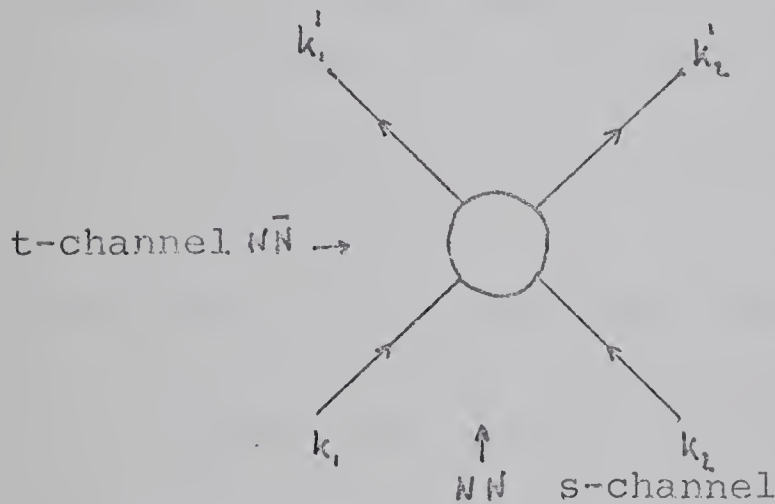


figure 41

The causal amplitude is an operator in the isotopic spin space also. We can therefore write it as a linear combination of I_+ and I_- . I_1 denotes the total isotopic spin of the nucleon-antinucleon system in the t-channel. (The reason for our interest in the t-channel is that the $N\bar{N} \rightarrow N\bar{N}$ scattering can be related to $N\pi \rightarrow N\pi$ scattering by crossing).

$$I_+ = \frac{1}{4} (1 - \vec{\tau}_N \cdot \vec{\tau}_{\bar{N}}) \quad (13a)$$

$$I_- = \frac{1}{4} (3 + \vec{\tau}_N \cdot \vec{\tau}_{\bar{N}}) \quad (13b)$$

$A(s, t, u)$ can be expanded as :

$$A(s,t,u) = \sum_{j=1}^5 \sum_{i=\pm} p_j^i(s,t,u) I_i P_j \quad (14)$$

We consider the behaviour of the coefficients $p_j^i(s,t,u)$ under the transformation $k_2 \leftrightarrow -k_2'$, which corresponds to $s \leftrightarrow u$. The effect of I_i on the transformation is most easily seen by writing its matrix elements for the t-channel scattering. I_+ and I_- project out respectively states which are symmetric and antisymmetric in the particle labels. So :

$$\langle k_1 -k_1' | I_{\pm} | k_2' -k_2 \rangle = \mp \langle k_1 -k_1' | I_{\pm} | -k_2 k_2' \rangle \quad (15)$$

The transformation properties of the kinematic invariants are slightly more involved. An explicit treatment is given in the Amati et. al. paper. The result is :

$$\langle k_1 -k_1' | P_j | k_2' -k_2 \rangle = (-1)^j \langle k_1 -k_1' | P_j | -k_2 k_2' \rangle \quad (16)$$

From (14) - (16) it follows that :

$$p_j^{\pm}(s,t,u) = \mp (-)^j p_j^{\pm}(u,t,s) \quad (17)$$

The transformation $s \leftrightarrow u$ is of interest because it transforms the box and crossed diagrams into each other. This is illustrated in figure 42.

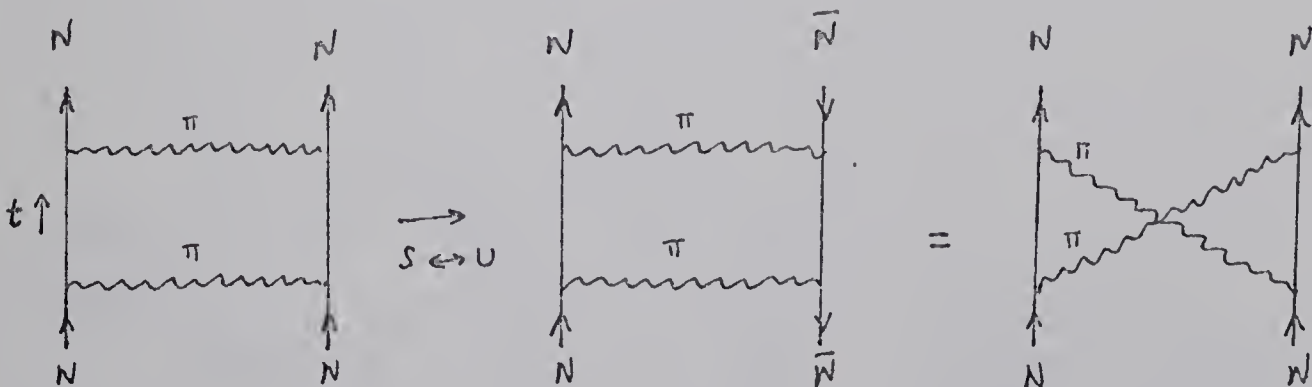


figure 42 : the transformation $s \leftrightarrow u$ of the box diagram into the crossed diagram

By calculation of the box diagram one can show that the coefficients of the invariant functions have the form (ref. 130) :

$$p_j^\pm(s, t, u) = \frac{1}{\Pi} \int_{(2m)^2}^{\infty} dt' \int_{(2M)^2}^{\infty} ds' \sigma_j^\pm(s', t') \{ (t' - t - i\epsilon) (s' - s - i\epsilon) \}^{-1} \quad (18)$$

Consequently the sum of box and crossed diagrams becomes:

$$p_j^\pm(s, t, u) = \frac{1}{\Pi} \int_{(2m)^2}^{\infty} dt' (\rho_j^\pm(s, t') \mp (-1)^j \rho_j^\pm(u, t') (t' - t - i\epsilon)^{-1}) \quad (19)$$

where :

$$\rho_j^\pm(x, t) = \int_{(2M)^2}^{\infty} dx' \sigma_j^\pm(x', t) (x' - x - i\epsilon)^{-1} ; \quad x \in \{s, u\} \quad (20)$$

We shall assume that the relation (19) also applies if pion-pion correlations and pion-nucleon rescattering effects are taken into account. The coefficients of the kinematic invariants do not satisfy a Mandelstam representation; this is due to the crossing behaviour of the kinematic invariants, which introduces kinematical singularities (ref. 130). Therefore Amati et. al. use a modified set of relativistically invariant operators. The coefficients of the amplitudes with respect to this set can be written as a sum of $p_j^\pm(s, t, u)$ multiplied by kinematical factors.

The spectral functions ρ_j^\pm can be related to the $\pi\pi \rightarrow N\bar{N}$ and the $\pi N \rightarrow \pi N$ processes as illustrated in figure 43.

$N\bar{N} \rightarrow \pi\pi$

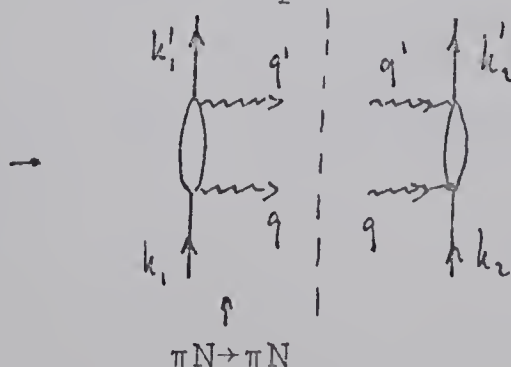


figure 43 :

relation between
 $NN \rightarrow NN$, $\pi N \rightarrow \pi N$ and
 $\pi\pi \rightarrow NN$

The Mandelstam variables for the $\pi N \rightarrow \pi N$ process are :

$$\begin{aligned} s_1 &= -(k_1 - q)^2 \\ t_1 &= -(k_1 - k')^2 = t \\ u_1 &= -(k_1 - q')^2 \end{aligned} \quad (21)$$

We briefly mention some relevant features of π -N scattering. There are two possible values for the total iso-spin quantum number, namely $\frac{1}{2}$ and $\frac{3}{2}$. Therefore if one assumes charge independence, one can distinguish two scattering amplitudes T_{\pm} , where the T_+ refers to the total iso-spin $\frac{3}{2}$, and T_- to the total iso-spin $\frac{1}{2}$. Due to the nucleon spin a further distinction between two amplitudes can be made. The most general form, given first by Chew et. al. (ref. 123), is :

$$T_{\pm}(s_1, t, u_1) = \mathcal{A}_{\pm}(s_1, t, u_1) - i \gamma^{\mu} Q_{\mu} \mathcal{B}_{\pm}(s_1, t, u_1) \quad (22)$$

where $Q = \frac{1}{2}(q' - q)$ and \mathcal{A}_{\pm} and \mathcal{B}_{\pm} are scalar functions of s_1, t and u_1 .

The crossing $s_1 \leftrightarrow u_1$ implies $Q \rightarrow -Q$ and $\Pi \rightarrow \bar{\Pi}$. One defines $\mathcal{A}^{\pm} = \frac{1}{2}(\mathcal{A}_{-} \pm \mathcal{A}_{+})$ and $\mathcal{B}^{\pm} = \frac{1}{2}(\mathcal{B}_{-} \pm \mathcal{B}_{+})$ which have the convenient crossing behaviour :

$$\begin{aligned} \mathcal{A}^{\pm}(s_1, t, u_1) &= \pm \mathcal{A}^{\pm}(u_1, t, s_1) \\ \mathcal{B}^{\pm}(s_1, t, u_1) &= \mp \mathcal{B}^{\pm}(u_1, t, s_1). \end{aligned} \quad (23)$$

Dispersion relations for the amplitudes \mathcal{A}^{\pm} and \mathcal{B}^{\pm} for a fixed value of t were given first by Chew et. al. (ref. 123). The signs that appear in these, follow from equations (23) :

$$\mathcal{A}^{\pm}(s_1, t, u_1) = \frac{1}{\Pi} \int_{(M+m)^2}^{\infty} ds'_1 \sigma_A^{\pm}(s'_1, t) [(s'_1 - s_1 - i\epsilon)^{-1} \pm (s'_1 - u_1 - i\epsilon)^{-1}] \quad (24a)$$

$$\mathcal{B}^{\pm}(s_1, t, u_1) = g^2 \left[\frac{1}{M^2 - s_1} \mp \frac{1}{M^2 - u_1} \right] + \frac{1}{\Pi} \int_{(M+m)^2}^{\infty} ds'_1 \sigma_B^{\pm}(s'_1, t) \cdot [(s'_1 - s_1 - i\epsilon)^{-1} \mp (s'_1 - u_1 - i\epsilon)^{-1}] \quad (24b)$$

These dispersion relations are valid only if the spectral functions drop fast enough for large values of s_1 . This condition is in fact not satisfied, and subtractions should be made. This question is not very important for our discussion. Details are given in reference 4. The pole terms represent the intermediate nucleon state. From the details of the spin structure of (22), one can show that these arise only in the dispersion relation for \mathcal{B} . A discussion of this is given in reference 62.

The relation of T^{\pm} with the nucleon-nucleon scattering amplitude A is found by generalizing the spinless result (112) of chapter II. Using the normalizations of Amati et. al., one finds :

$$\langle N\bar{N} | \text{Im}A | N\bar{N} \rangle = \frac{1}{2(2\Pi)^2 t} S_{\pi\pi} \langle N\bar{N} | T^{\dagger} | \pi\pi \rangle \langle \pi\pi | T | N\bar{N} \rangle \quad (25)$$

where $S_{\pi\pi}$ denotes the summation (integration) over all intermediate two pion states and :

$$\langle \pi_{\alpha} \pi_{\beta} | T | N\bar{N} \rangle = T^{+} \delta_{\alpha\beta} + \frac{1}{2} [\vec{\tau}_{\beta}, \vec{\tau}_{\alpha}] T^{-} \quad (26)$$

T^{+} and T^{-} denote the amplitudes with definite total iso-spin 0 and 1 respectively in the t -channel. α and β denote the iso-spin indices of both pions. Substitution of (26) into (25) yields after a few steps :

$$\langle N\bar{N} | \text{Im}A | N\bar{N} \rangle = 3 \langle N\bar{N} | \text{Im}A^{+} | N\bar{N} \rangle + 2 \langle N\bar{N} | \text{Im}A^{-} \vec{\tau}_1 \cdot \vec{\tau}_2 | N\bar{N} \rangle \quad (27)$$

where A^+ and A^- are amplitudes with a definite total iso-spin 0 and 1 in the t-channel (ref. 125).

The relations (14), (19), (21) and (25) - (27) enable one to write the spectral functions ρ^\pm in terms of the amplitudes \mathcal{A}^\pm and \mathcal{B}^\pm . The resulting expressions are integral equations over the intermediate pion momenta. Explicit expressions of these and their solution can be found in the textbook of Brown and Jackson (ref. 4). This formalism enables one to use $\pi\pi\rightarrow N\bar{N}$ and $\pi N\rightarrow\pi N$ data in establishing the two pion exchange effect in the nucleon-nucleon interaction. Experimentally much more is known about the pion-nucleon scattering process than about the $\pi\pi\rightarrow N\bar{N}$ process. A problem connected with the use of pion-nucleon data is that the calculations of the invariant functions $p_j^\pm(s,t,u)$ require knowledge of the spectral function ρ_j^\pm for $t\geq 4m^2$. However the physical region for pion-nucleon scattering is $t\leq 0$. Therefore to use pion-nucleon data, one has to make an extrapolation to $t\geq 4m^2$. This is done by expanding the spectral functions σ_A^\pm and σ_B^\pm of (24a) and (24b) in Legendre polynomials. The relevant angle in this expansion is given by :

$$\cos\phi = 1 + \frac{t}{2q^2} \quad (28)$$

ϕ and q are the scattering angle and 3-momentum magnitude respectively in the cm-system for $\pi N\rightarrow\pi N$ scattering (ref. 4). From (28) it is clear that the Legendre polynomial expansion is essentially an expansion in powers of t . If the spectral functions σ_A^\pm and σ_B^\pm depend only weakly on t , one may hope that the extrapolation to $t\geq 4m^2$ is allowed. There are strong correlations between the pions in the s - and p-states. Therefore for these partial waves, one can not expect that the extrapolation is justified.

However as no low energy pion-pion resonances are known for pion-pion partial waves for which $l \geq 2$, one may hope that for these partial waves the extrapolation can be done. Assuming this to be the case, Amati et. al. (ref. 130) suggested the following treatment :
Write the spectral functions ρ_j^\pm as the sum of three contributions :

$$\rho_j^\pm = d_j^\pm - b_j^\pm + c_j^\pm \quad (29)$$

where d_j^\pm is the contribution obtained from the dispersion relations (24a) and (24b). Assuming that σ_A^\pm and σ_B^\pm are analytic in the complex t -plane. This analyticity condition means that one supposes that no strong pion-pion correlations exist. Of course this is certainly not justified for the s - and p -waves. b_j^\pm represents the s - and p -wave projections of d_j^\pm which should be subtracted. c_j^\pm represents the total contribution of the pion-pion s - and p -waves. Information of these contributions can be obtained from the helicity amplitudes f_+^0 and f_\pm^1 for the reaction $\pi\pi \rightarrow \bar{N}N$. These amplitudes were introduced first by Frazer and Fulco (ref. 125). The subscript $+$ denotes the process in which the helicities of nucleon and antinucleon are equal, the $-$ refers to opposite helicities. The 0 and 1 denotes the s - and p -wave respectively. As a consequence of the Pauli principle the contribution of f_-^0 vanishes (ref. 125). Chemtob et. al. (ref. 109) give explicit expressions of c_j^\pm in terms of f_+^0 and f_\pm^1 . In the approach, that we outlined, the TPE-contribution is entirely determined by the quantities $\sigma_A^\pm, \sigma_B^\pm$ and f_\pm^0, f_\pm^1 .

Chemtob et. al. define a TPE-potential by observing that :

$$A = \sum_{j=1}^5 \{ 3p_j^+ + 2p_j^- \vec{\tau}_1 \cdot \vec{\tau}_2 \} P_j \quad (30)$$

(this follows from (27)), which suggests a similar expansion of $U^{(4)}$:

$$U^{(4)} = \sum_{j=1}^5 \{ 3v_j^+ + 2v_j^- \vec{\tau}_1 \cdot \vec{\tau}_2 \} P_j \quad (31)$$

Analogous to relation 6, one may equate :

$$v_j^i = p_j^i - r_j^i \quad (32)$$

where r_j^i denotes the contribution from the iterated one pion exchange. They show that for r_j^i a similar spectral representation as for p_j^i holds. The spectral function in this expression is calculated. This calculation is in much detail repeated in the textbook of Brown and Jackson (ref.4).

Chemtob et. al. considered a model that includes the one and two pion exchange, the ω and the η -meson.

The spectral functions σ_A^\pm and σ_B^\pm were approximated by the contributions of the three lightest nucleon iso-bars $\Delta(1236)$, $N(1470)$ and $N(1520)$, which were treated as stable particles. Then the spectral functions have the form of a sum of δ -functions :

$$\begin{pmatrix} \sigma_A^\pm(s_1, t) \\ \sigma_B^\pm(s_1, t) \end{pmatrix} = \sum_i \begin{pmatrix} G_{A_i}^\pm(t) \\ G_{B_i}^\pm(t) \end{pmatrix} \delta(s_1 - M_i^{*2}) \quad (33)$$

In (33) i denotes the nucleon iso-bar and M_i^* its mass. The t -dependence of the residues is at most linear for the spin- $\frac{3}{2}$ resonances $\Delta(1236)$ and $N(1520)$. For the spin- $\frac{1}{2}$ iso-bar $N(1470)$, no t -dependence is allowed.

Experimentally the helicity amplitude f_+^0 was not too well-known. Several descriptions were used by various authors, which did not always agree (ref. 126). It is

interesting that recently, progress in this respect seems to be made (ref. 127) by using more sophisticated methods of analytic continuation from pion-nucleon and pion-pion scattering amplitudes. Chemtob et. al. used a form essentially based on soft pion arguments. More knowledge is available about the f_{\pm}^1 amplitudes, because additional information is provided by the iso-vector form factors of the nucleons (ref. 126). Different methods of determining f_{\pm}^1 yield similar results. The ρ -resonance was included by assuming a Breit-Wigner form.

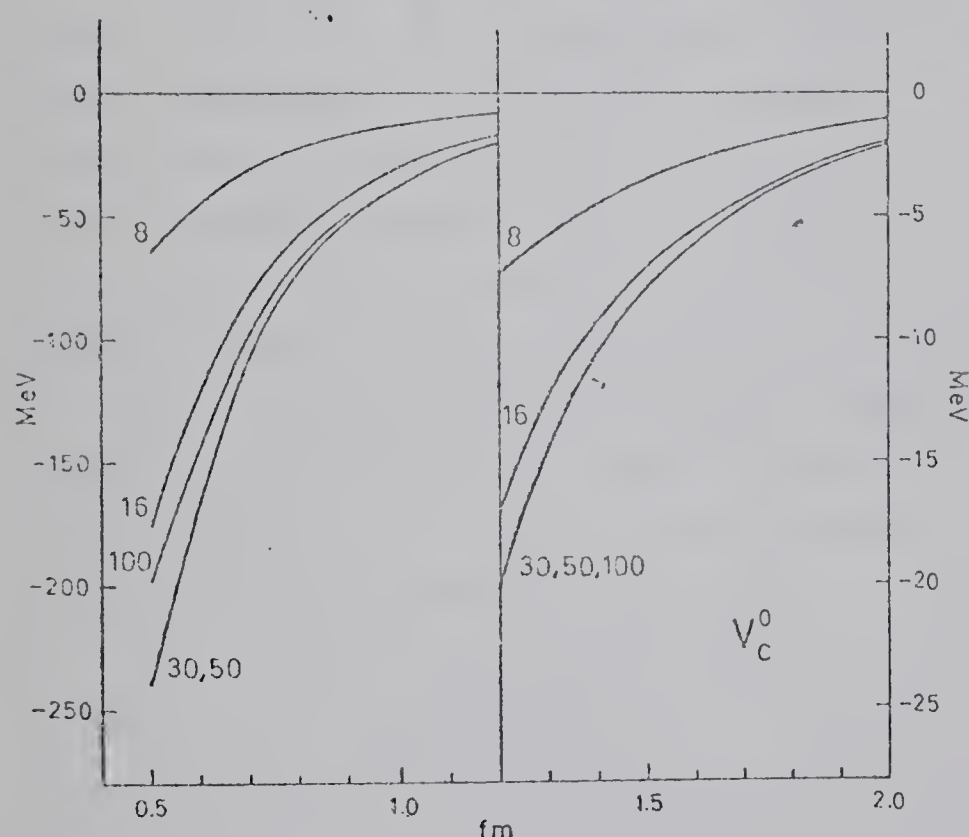


figure 44 :
the iso-singlet
central potential
for different va-
lues of the cut-
off constant
 t_{\max} ; each curve
is labeled by the
 t_{\max} value in
units of m_{π}^2

An adiabatic reduction of the resulting expression for the momentum space potential was used to calculate a coordinate space representation. All parameters were pre-determined. A problem Chemtob et. al. encountered, was the divergence of the t -integration, which was regularized by introducing a cut-off. However the inclusion of the so called rescattering effects described by the spectral functions

σ_A^\pm and σ_B^\pm , introduced a critical dependence of the resulting potential on the cut-off value that was used. This undesirable feature was especially significant for small distances. They restricted their considerations to the region outside 0.5fm. It turned out that the potential was generally in agreement with the Hamada-Johnston potential, in particular at distances larger than 0.8fm.

Figure 44 on the previous page, shows the dependence of the iso-singlet central potential to different values of the cut-off t_{\max} .

Cottingham and Vinh Mau and their Paris group (ref. 128 and 129), performed similar calculations. The main difference is that the spectral functions σ_A^\pm and σ_B^\pm were not approximated by the nucleon iso-bar contributions, but were determined from the experimental phase-shifts of the pion-nucleon scattering. This was done by adjusting the coefficients in the power series expansion of σ_A^\pm and σ_B^\pm in t to the empirical values. Furthermore subtracted dispersion relations were used, suggested by the Regge asymptotic behaviour of the cross-sections. Details of these Regge theory arguments are given in the references 128 and 4. Their model included besides the one and two pion exchange the ω -contribution. A coordinate space potential was calculated for distances larger than 0.6fm, and a quantitative fit to the Hamada-Johnston potentials was established. It is interesting that this is possible without adjustable parameters. In a second article (ref. 129), phase parameters were calculated. These were determined directly from the amplitude without going through a potential. The discussion was restricted to partial waves for which $J \geq 2$. For these one can apply a simple unitarization correction, such as the K-matrix method. A reason for the restriction to $J \geq 2$ is, that for higher partial waves the phase parameters are not very sensitive to the cut-off

value t_{\max} . This can be seen from the following form, in which spin and iso-spin are suppressed for simplicity :

$$A_L(s) = \int_{-1}^1 dz P_L(z) \frac{1}{\Pi} \int_{4m^2}^{t_{\max}} dt' \rho(t', s) (t' - t - i\epsilon)^{-1} \quad (34)$$

$$z = 1 + \frac{t}{2k^2} ; k = |\vec{k}|$$

$$= \frac{1}{k^2} \int_{4m^2}^{t_{\max}} dt' \rho(t', s) Q_L\left(1 + \frac{t'}{2k^2}\right) \quad (35)$$

\vec{k} is the 3-momentum of a nucleon in the s-channel cm-system and Q_L is the Legendre polynomial of the second kind. The interesting observation one can make is that $Q_L\left(1 + \frac{t'}{2k^2}\right)$ decreases rapidly as a function of t' for $L > 2$, thereby making $A_L(s)$ less dependent on t_{\max} .

A satisfactory agreement with a Livermore phase-shift analysis (ref. 98) could be obtained, considering that no adjustable parameters were used. The figures 45 and 46 illustrate their results for the ϵ_2 and 3F_3 phase-shifts.

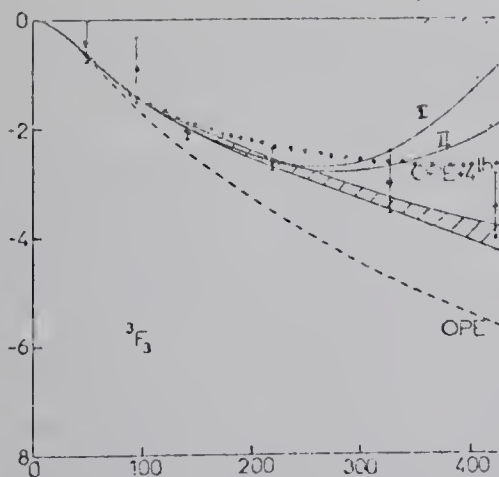


figure 45

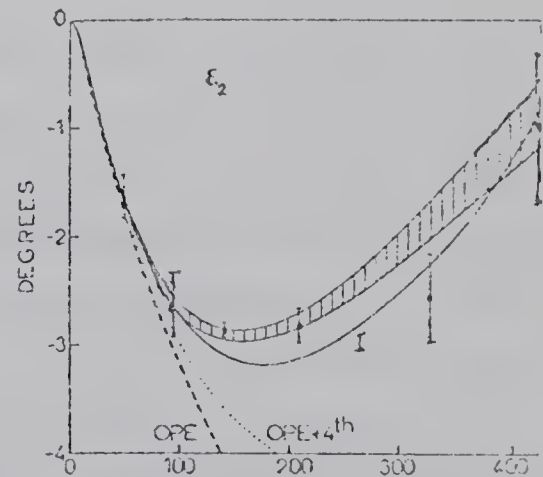
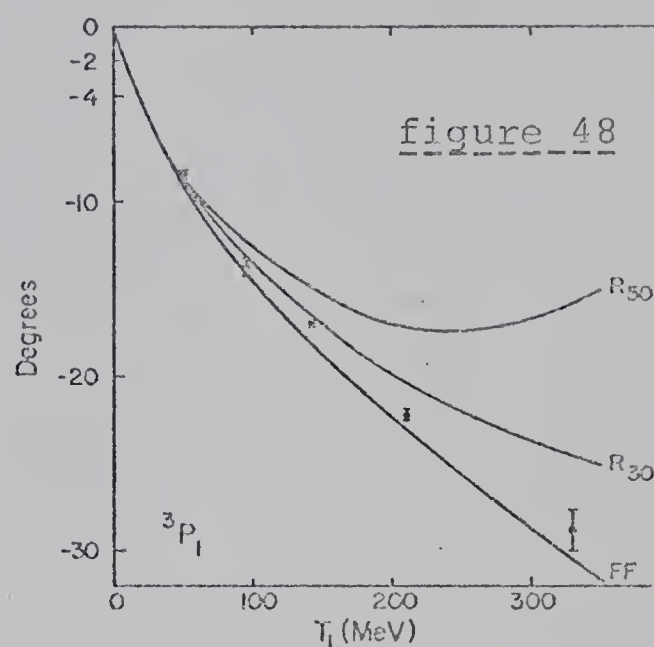
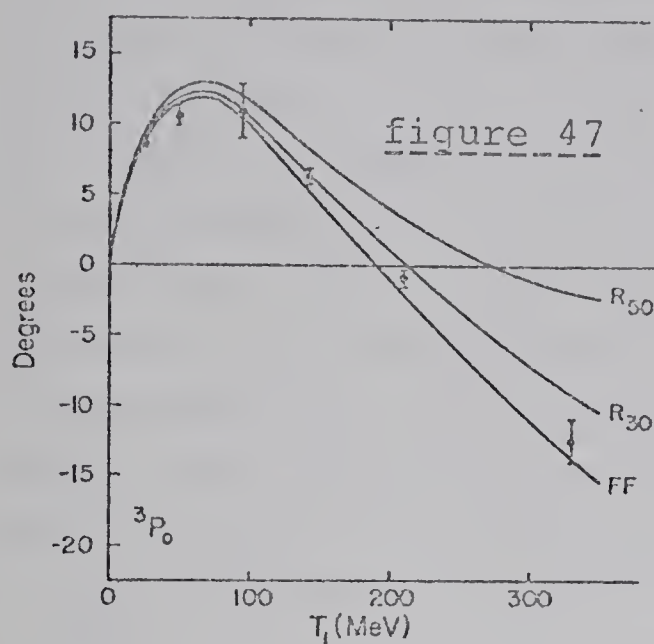


figure 46

the ϵ_2 (fig. 45) and 3F_3 (fig. 46) phase parameters calculated by Vinh Mau et. al. (ref. 129); the curves I and II show the results for two different phase analyses; furthermore the results for the OPE and the OPE plus the fourth order contribution are shown; the shaded area and the points show the energy dependent - and the energy independent Livermore phase analysis (ref. 98)

The sensitivity of the results for low partial waves to the value of the cut-off, is illustrated in the figures 47 and 48. These are results obtained by Jackson et. al. (ref. 126).



phase-shifts for the 3P_0 (fig. 47) and the 3P_1 (fig. 48) partial waves, calculated by Jackson et. al.; the curves R_{30} and R_{50} show the results for the cut-off values 30 m_π^2 and 50 m_π^2 ; in the curve denoted by FF no rescattering effects are included

In these results the corrections for $l \geq 2$ pion-pion partial waves is approximated by taking into account the contributions from the three lightest nucleon iso-bars. The conclusion made by them is that the extrapolation involved in relating the pion-nucleon scattering amplitude to the nucleon-nucleon scattering process, is not justified. An exception are the NN higher partial waves, because it follows from (35) that then $\rho(t', s)$ contributes only for low values of t' . In the low t' region the spectral functions σ_A^\pm and σ_B^\pm are known to depend only weakly on t' for $l \geq 2$.

Jackson et. al. considered a number of models in which the rescattering effects for $l \geq 2$ are left out entirely. This is not unreasonable, as one expects in general the largest contributions from the lowest partial waves.

In this approximation only the nucleon pole term contributes to d_j . Satisfactory results can be obtained in this way, as is seen by comparing the 3P_0 and 3P_1 phase-shifts with and without the rescattering effects. This is shown in the figures 47 and 48 on the previous page. In the various models that Jackson et. al. discussed, different choices for f_+^0 and f_{\pm}^1 are compared. The results turned out to be rather sensitive on f_+^0 . This was considered as an unfortunate result in view of the lack of knowledge about this amplitude (Recently improvements were made (ref. 127)). Figures 49 and 50 show the results for the 1P_1 and 3D_1 phase-shift.

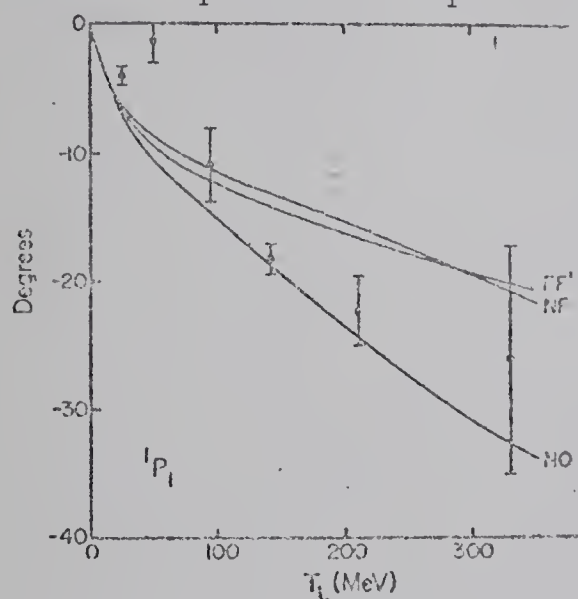


figure 49

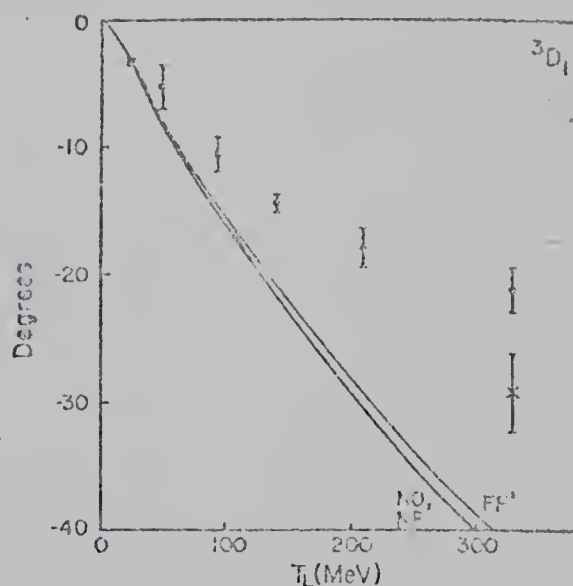


figure 50

phase-shifts for the 1P_1 (fig. 49) and 3D_1 (fig. 50) partial waves, calculated by Jackson et. al.; NO, FF' and NF denote different models for f_+^0 and f_{\pm}^1

One notices that the 3D_1 phase-shift is not well-reproduced. However the other phase parameters are in reasonable agreement with the empirical values. The figures give the results for various choices of the amplitudes f_+^0 and f_{\pm}^1 . Little is known about these amplitudes for t -values larger than $50 m_{\pi}^2$. Fortunately one may hope that the contributions for larger values of t are

relatively unimportant since these correspond to small distances between the nucleons, a region which is dominated by the strong repulsion, obtained from the ω -meson exchange. In practice therefore one applies a cut-off with a typical value of $50 m_\pi^2$. Jackson et. al. studied the cut-off dependence of the phase-shifts by using different values for t_{\max} in the range $40 m_\pi^2$ - $60 m_\pi^2$. Without the $\ell \geq 2$ rescattering effects a nearly energy independent correction of about 2 degrees for the S-wave phase-shift was found, which could be compensated by a variation of the ω -N coupling constant, within its range of experimental uncertainty. The dependence of the other phase-shifts was quite negligible.

5) Form Factors

In order to regularize the Blanckenbecler-Sugar equation, form factors have to be used. Jackson et.al. (ref. 126) considered the familiar dipole form. This form was compared with a more sophisticated cut-off factor based on multiple neutral vector meson processes in the relativistic eikonal approximation. This form is (ref. 131):

$$\exp(2i\{[\chi(t) - \chi(m_i^2)] + [\chi(u) - \chi(4M^2 - s - m_i^2)]\}) \quad (36a)$$

where :

$$i\chi(x) = \begin{cases} -2\gamma(2M^2 - x) \{x(4M^2 - x)\}^{-1/2} \arctan\{x(4M^2 - x)^{-1}\} & \text{for } 0 < x < 4M^2 \\ -2\gamma(2M^2 - x) \{-x(4M^2 - x)\}^{-1/2} \ln\left\{\left(\frac{-x}{4M^2}\right)^{1/2} + \left(1 - \frac{x}{4M^2}\right)^{1/2}\right\} & \text{for } x < 0 \end{cases} \quad (36b)$$

This form factor is normalized so that on-shell the residues of the pole terms are unchanged. γ is a parameter. The factor $\chi(t)$ arises from the diagrams shown in figure 51a. The contributions that lead to the factor $\chi(u)$ are shown in figure 51b.

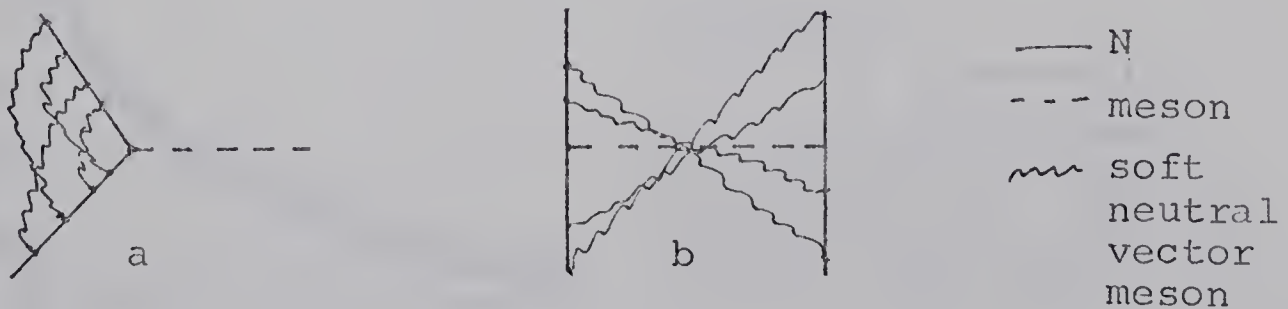


figure 51 : diagrams representing contributions to the eikonal form factor

The essential assumption that leads to the form (36a) and (36b) is that the momentum transfer at the nucleon neutral vector meson vertices is small compared to the momentum transferred by mesons in the model under consideration. One can show that by using pseudo-scalar mesons, instead of neutral vector mesons, the resulting expression vanishes (of course using ps-ps coupling). Problems arise if we want to use this idea for charged vector mesons. Due to the iso-vector algebra one can not apply the techniques used for the neutral vector mesons. The effects of the diagrams 51a and 51b for charged vector mesons are in fact not known (ref.4). The restriction to neutral vector mesons is rather arbitrary. However the form is quantitatively successful as a model for the electro-magnetic form factor (ref. 121), which gives it some credibility for use in the strong interaction. The data in the electro-magnetic case suggest a value $1.25 < \gamma < 1.50$ (ref. 89). Jackson et. al. compared a fit to the S-wave phase-shift using the form (36a) and (36b) to a fit in which the dipole form for the cut-off factor is used. In the dipole form the value $1.54 M$ for the cut-off mass was chosen, which is in agreement with the electro-magnetic form factor data. Figure 52 shows the result for the 3S_1 phase-shift.

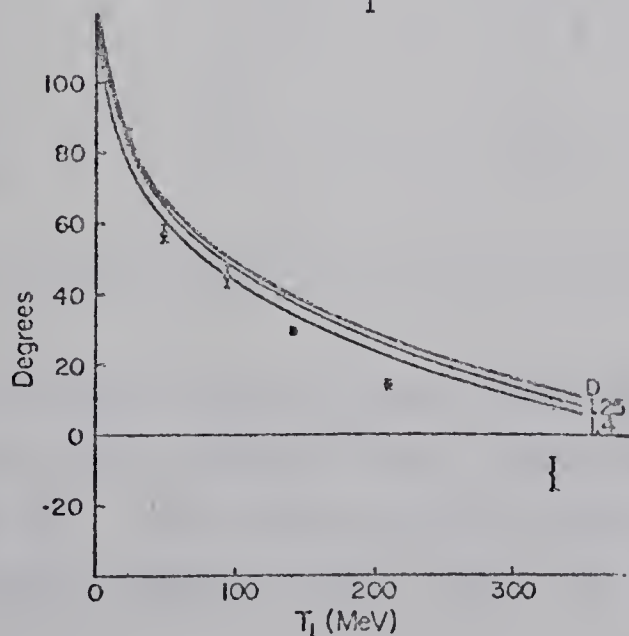


figure 52 :
the 3S_1 phase-shift
calculated by Jackson
et. al. using a dipole
form factor (D) and the
eikonal form factor for
 $\gamma=1.25$ and $\gamma=1.4$

One notices that the phase-shift is not sensitive to the particular form of the cut-off factor that is chosen, as long as these are in agreement with the electro-magnetic form factor data. Although the form is not very critical to obtain a good fit, the coupling constants in OBE-models using the form (36 a,b) are much closer to their expected values. This feature comes about as the exponential form dampens the high momentum transfers more effectively than a dipole form. The dipole form influences also the low momenta, forcing the use of higher values for the coupling-constants. This was observed by Holinde and Machleidt (ref. 131), who did an OBE-analysis using the eikonal form factor. A reduced χ^2 of 2.77 could be obtained for 132 data points using the coupling constants shown in table 14.

<u>table 14</u> (ref. 131)				
<u>meson</u>	g^2 <u>eikonal</u>	g^2 <u>dipole</u>	f/g <u>eikonal</u>	f/g <u>dipole</u>
π	14.1	14.1		
η	2	2.2		
σ	5.66	5.9		
δ	0.82	8.7		
ρ	0.5	1.4	6.2	4.5
ω	10	25		
ϕ	0	33.6		

The first column gives the values of the coupling constants using the eikonal form factor and the electro-magnetic value $\gamma=1.25$, and the second column gives the value of their earlier model (ref. 146), in which the dipole cut-off

factor was used. The meson masses were the same in both models, except for the σ and ρ . The masses $m_\sigma = 520$ MeV. and $m_\rho = 711$ MeV. were used instead of $m_\sigma = 500$ MeV. and $m_\rho = 763$ MeV. of the earlier model. The use of an effective ρ -meson mass is a well-known method to account for the large width of the ρ (ref. 131).

6) Concluding Remarks; Other Higher Order Effects

In the foregoing we considered the contributions of the two pion exchange and the nucleon iso-bars. Both turned out to be important effects. The results of the zero-parameter models of Vinh Mau et. al. (ref. 129) and Jackson et. al. (ref. 126) show a qualitative agreement with the experimental data. However the model of Vinh Mau et.al., which includes rescattering effects for the $l \geq 2$ waves, can only be used for the higher NN partial waves. Otherwise the results are strongly dependent on the cut-off that is used. Jackson et. al. neglect these rescattering effects and are able to calculate the phase parameters for all partial waves. Although quantitatively the results agree with the data, a quantitative fit is not possible. Especially the 3D_1 phase-shift remains a problem.

The OBE-model of Holinde et. al. (ref. 141), augmented with effects from the intermediate $\Delta(1232)$, gives a poor description of the inner region. The results depend strongly on the cut-off mass. Furthermore the eikonal form

appeared inadequate.

The necessity of a cut-off mechanism was already clear from the calculations of Sugawa and von Hippel (ref. 143). They were first to consider the transition potential for the Δ -excitation due to a π -exchange. They noticed that this transition potential develops a singular r^{-3} behaviour.

It should be very satisfactory if one could find a physical process which provides an effective cut-off.

Green and Haapakoski (ref. 144) considered the Δ -excitation due to the exchange of a ρ . They showed from quark model considerations that the transition potential in this process differs in sign from the transition potential, which arises in the case of pion exchange. Important cancellations are therefore expected between the contributions from the two diagrams, shown in figure 53.

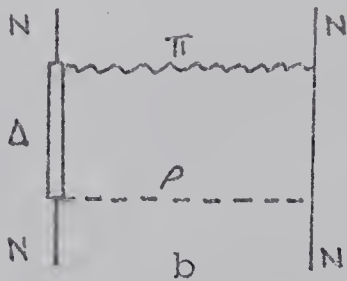
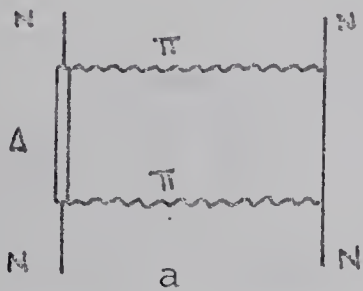


figure 53 :
diagrams con-
sidered by
Green and
Haapakoski
(ref. 144)

The potentials due to the processes represented by these diagrams are shown in figure 54.

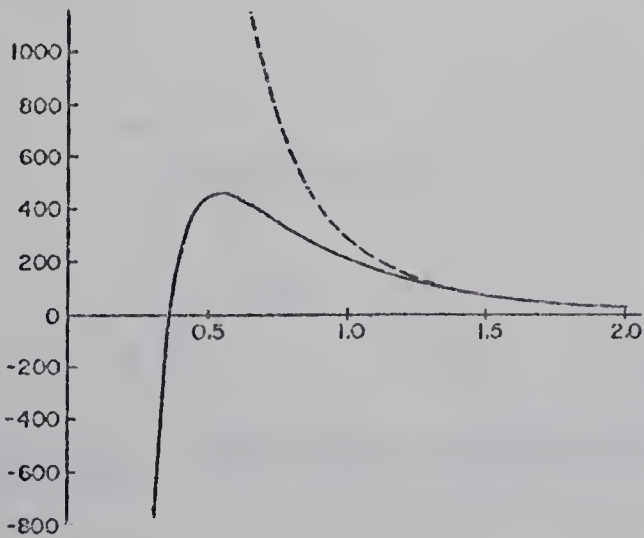


figure 54 :
the potentials cor-
responding to the
diagrams 53a (dashed
curve) and 53b (solid
curve)

Therefore the inclusion of the combined $\pi\rho$ exchange is quite crucial, one may hope that the $\Delta(1232)$ contribution becomes much less cut-off dependent. One may observe that the iso-scalar mesons, such as the η and ω , give no contribution to the transition potential.

Other processes that can be of importance are the corrections to the πN vertex due to the large width of the ρ -meson, which is obviously not included in the eikonal form factor. This is shown in figure 55. Jackson et. al (ref. 126) suggest that processes like these may result in a better fit of the D-phase-shift.

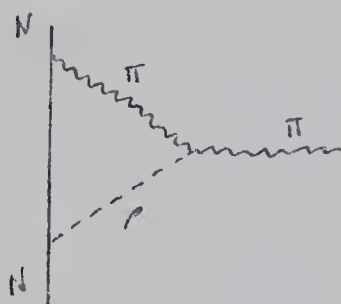


figure 55 :
 πN vertex
correction

A number of authors have studied the simultaneous exchange of the π and the η (ref. 1). They concluded that the effects are negligible, as could have been anticipated because of the small coupling between the nucleon and the η .

Recently Riska (ref. 145) has studied the exchange of a π together with the ω . He calculated the diagrams :

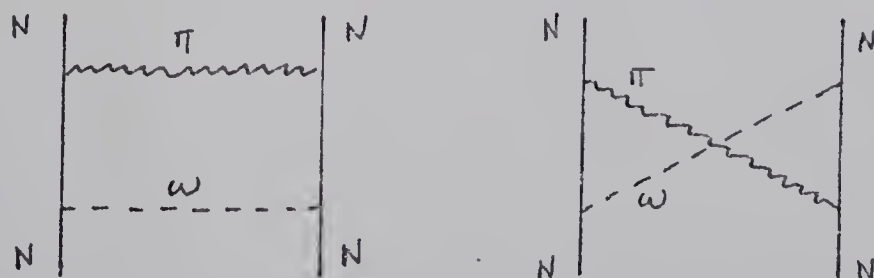


figure 56 : diagrams calculated by Riska (ref. 145)

Essentially the same formalism that Chemtob et. al (ref. 109) used to their calculation of the two pion exchange potential was applied. He calculated a coordinate space potential, which involved a non-relativistic reduction. The expressions that he obtains are integrations over t which converge slowly. A cut-off was introduced, as the integrand was not reliable for high values of t , in view of the non-relativistic reduction. Unfortunately this introduced a significant cut-off dependence of the resulting potential, especially for distances below 0.6fm, for which the potential is most important. The main result was that the tensor - and spin-spin forces are of the same order of magnitude as those of the single ω -exchange. This result is independent of the cut-off factor.

Finally we should mention that no serious attempt to calculate the uncorrelated three pion exchange contribution has appeared in the literature. Although the calculation of a number of 3π exchange diagrams is in progress. In particular one hopes to account for the D-phase-shifts by including 3π exchange effects (ref. 147):

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